A Coq formalization of digital filters

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Abstract. Digital filters are small iterative algorithms, used as basic bricks in signal processing (filters) and control theory (controllers). They receive as input a stream of values, and output another stream of values, computed from their internal state and from the previous inputs. These systems can be found in communication, aeronautics, automotive, robotics, etc. As the application domain may be critical, we aim at providing a formal guarantee of the good behavior of these algorithms in time-domain. In particular, we formally proved in Coq some error analysis theorems about digital filters, namely the Worst-Case Peak Gain theorem and the existence of a filter characterizing the difference between the exact filter and the implemented one. Moreover, the digital signal processing literature provides us with many equivalent algorithms, called realizations. We formally defined and proved the equivalence of several realizations (Direct Forms and State-Space).

1 Introduction

Most embedded systems, from planes to MP3 players, rely on numerical signal processing filters. Such a filter takes as inputs an infinite sequence of values, such as measurements of sensors, and returns an infinite sequence of values, such as a sound to be played or a value to control a nuclear power plant (see also §2.2). Applications of digital filters are therefore numerous, from trivial to life-critical, and formal methods have already been applied to such systems.

Digital filters have previously been formalized in HOL [1]: Akbarpour and Tahar define filters and somehow define the error filter as done in §5.1. But they do not bound the output error, while we do it using the Worst-Case Peak Gain (WCPG) Theorem of §5.2. We indeed benefit from the most recent advances for filter error bounding [2]. Another recent work in HOL by Siddique *et al.* focuses on the frequency-domain analysis and the z-Transform [3]. They consider causal filters and difference equations with definitions similar to ours. But their analysis is complementary to ours as we choose a time-domain analysis in order to next focus on finite-precision implementations. Akbarpour *et al.* previously compared floating-point and fixed-point implementations of digital filters to ensure their

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similar behavior [4]. But they require many hypotheses, including the absence of overflow in the computations. Park *et al.* aim at specifying filters and then proving their correctness [5], assuming that all the computations are exact. They then relax this assumption, but bound the floating-point error for one iteration step only, while the difficult part is the propagation of these errors [6]. Last, Feret developed a specific abstract domain for digital filters [7]. Finally, references in digital signal processing can be found in §2.

This paper presents a formalization of digital filters in the Coq proof assistant [8,9].[5] We rely on the standard library for defining reals and on the Coquelicot library [10] for real analysis and for the Limited Principle of Omniscience (LPO) which is derived from the axiomatization of reals. In addition, we use two axioms. Firstly, Functional Extensionality states that two functions sharing the same value on every input are equal. Secondly, Proof Irrelevance means that two proofs of the same property are equal. These axioms are often used to handle functions and proof objects in a more natural way and are considered safe.

The goals of this paper are a formal definition of a digital filter, its various algorithms and their equivalences, and some formal definitions and proofs about the error analysis to be used for fixed-point and floating-point implementations. This paper is organized as follows. Section §2 gives some background on signal processing and numerical filters. Section §3 presents our formalization choices. Section §4 defines some of the many (mathematically) equivalent algorithms, called realizations, that are used to describe a filter in signal processing. We also prove their equivalences, so that results established for one of them also hold for the others. Section §5 proves the error filter decomposition and the Worst-Case Peak-Gain theorem. Together, these two results allow to bound the final impact of adding a bounded error term at each computation step, having taken into account the propagation of these errors as each of them affects every future step. We were able to formalize this using only real numbers as there is no need to know where the error terms come from, but of course this is intended to be applied to rounding errors in finite-precision arithmetic. Finally, Section §6 concludes and gives some perspectives.

Notation: throughout the article, matrices are in uppercase boldface (e.g. $\boldsymbol{A}$), vectors are in lowercase boldface (e.g. $\boldsymbol{a}$), scalars are in lowercase (e.g. a). The matrix $\boldsymbol{I}_n$ is the identity matrix of size n.

2 Digital filters

2.1 Signals and operations

A discrete-time signal x is an ordered sequence of numbers denoted $x(k)$, where k is an integer. In a practical setting, such sequences often arise from periodic sampling of continuous time signals $x_c(t)$ where t represents time.

The signal x can be defined for any integer k or for some finite contiguous set of integers. We here restrict k to be in $\mathbb{N}$, or more precisely k to be in $\mathbb{Z}$, but with $x(k) = 0$ for $k < 0$. A signal can be real ($x(k) \in \mathbb{R}$) or vector ($\boldsymbol{x}(k) \in \mathbb{R}^{p\times 1}$).

[5] Available at `www.lri.fr/~gallois/code/coq-digital-filters-CICM18.tgz`

The simplest signal is probably the *impulse* signal (also called *Dirac* signal), denoted δ and defined as $\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{elsewhere.} \end{cases}$

This signal is central in linear signal processing theory since any signal can be expressed as an infinite sum of impulse signals (see §2.3).
Three elementary operations on signals can be defined:

– Addition: The sum of two signals x_1 and x_2 is their term-by-term sum, i.e. $y = x_1 + x_2$ means $\forall k,\ y(k) = x_1(k) + x_2(k)$.
– Scaling: $y = \alpha x$ is the sequence x scaled by $\alpha \in \mathbb{R}$, i.e. $\forall k,\ y(k) = \alpha x(k)$.
– Time shifting or delay: Let y be the sequence x shifted in time by an integer $K \geq 0$, then $\forall k,\ y(k) = \begin{cases} x(k-K) & \text{if } k \geq K \\ 0 & \text{elsewhere.} \end{cases}$

2.2 Linear Time Invariant filters

A filter $\mathscr{H}$ is a mathematical transformation that maps an input signal u into an output signal $y = \mathscr{H}\{u\}$, as shown in Figure 1. At each time k, the filter produces an output $y(k)$. But, contrary to usual mathematical functions, the output $y(k)$ depends not only on the input $u(k)$, but also on the internal state of the filter (i.e. on the initial condition of the filter and the previous inputs).

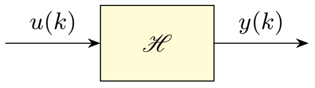


Fig. 1: A discrete-time filter.

A linear time invariant (LTI) filter satisfies the two following properties:

– Linearity: The filter is linear with respect to its input, i.e.

$$\mathscr{H}\{\alpha u_1 + \beta u_2\} = \alpha \mathscr{H}\{u_1\} + \beta \mathscr{H}\{u_2\} \tag{1}$$

– Shift invariance: if the input of the filter is delayed by $K \geq 0$ samples, then the output is also delayed by K samples, i.e. if $\forall k, u_1(k) = u_2(k-K)$, then

$$\forall k, \quad \mathscr{H}\{u_1\}(k) = \begin{cases} \mathscr{H}\{u_2\}(k-K) & \text{if } k \geq K \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The filter can have as inputs and outputs either scalars (Single-Input Single-Output filter, aka SISO filter), or vectors (Multiple-Input Multiple-Output filter, aka MIMO filter). Due to the linearity of LTI filters, a q-input p-output MIMO filter can be seen as the assembly of $p \times q$ SISO filters, where the ij^{th} element is the filter that captures the effect of the j^{th} input on the i^{th} output.

LTI filters are compositions of the three elementary operations on signals (addition/subtraction, multiplication by constant and delay, the last one being classically denoted z^{-1} as in the z-transform [11,12]). Data-flow graphs using these operations as blocks and signal as streams are widely used in signal and control theory to describe the different realizations, as shown in Figures 3a, 3b, and 4a.

2.3 Impulse response

The impulse response of a SISO filter $\mathscr{H}$, denoted h, is the answer (output) of the filter to an impulse input δ (i.e. $h = \mathscr{H}\{\delta\}$).

Since any input u can be written as an infinite sum of weighted shifted impulses $u(k) = \sum_{l\in\mathbb{Z}} u(l)\delta(k-l)$, then, by linearity of the considered filter, the output y of u through $\mathscr{H}$ can be obtained by

$$y(k) = \mathscr{H}\{u\}(k) = \sum_{l\in\mathbb{Z}} u(l)h(k-l) \tag{3}$$

The output y is obtained as a convolution of signals u and h. For that reason, the impulse response of a filter fully defines it. From the impulse response characterization, the LTI filters can be divided in two types: the Finite Impulse Response (FIR) filters, where h is null above a certain time, and the Infinite Impulse Response (IIR) filters, where h has infinite support [11].

2.4 Constant-Coefficient Difference Equation

An important subclass of LTI filters consists of those for which a n-order constant-coefficient difference equation exists between inputs and outputs, i.e. the last n inputs and outputs are linked by $\forall k, \sum_{i=0}^{n} a_i y(k-i) = \sum_{i=0}^{n} b_i u(k-i)$, where the $\{a_i\}_{0\leq i\leq n}$ and $\{b_i\}_{0\leq i\leq n}$ are constant coefficients.

We also assume that $a_0 = 1$, so that the equation is rearranged as

$$y(k) = \sum_{i=0}^{n} b_i u(k-i) - \sum_{i=1}^{n} a_i y(k-i). \tag{4}$$

This relationship describes an IIR filter as soon as the a_i's coefficients are not all null, otherwise it describes an FIR filter. The value n is said to be the *order* of the filter. These coefficients are also the coefficients of the *transfer function* of the filter: this mathematical object describes its input-output relationship in frequency domain (whereas (4) describes it in time domain). It has also been formalized in Coq, but with few properties, and thus not presented here.

3 Formalization

3.1 Signals and filters

The first components required to work with digital filters are signals, presented in §2.1. For now, we consider real signals (sequences of real numbers). We define them as functions from $\mathbb{Z}$ to $\mathbb{R}$ that take the value 0 for every $k < 0$.

```
Definition causal (x : Z → R) := (forall k : Z, (k < 0)%Z → x k = 0%R).
Record signal := { signal_val :> Z → R ; signal_prop : causal signal_val }.
```

Axioms `FunctionalExtensionality` and `ProofIrrelevance`, discussed in the introduction, ensure that signals are fully characterized by their values for $k \geq 0$.

This definition of signals makes them easy to build recursively. Notably, order-n recursion (when $x(k)$ is built from $x(k-1), x(k-2), ..., x(k-n)$) is very common when working with filters: for example, (4) is an order-n recursive relation on y. For signals, the negative values are known (since they are zero), so we may use an order-n recursion. See §3.2 for more on recursive constructions.

We could have defined signals as functions from $\mathbb{N}$ to $\mathbb{R}$. Ironically, working with $\mathbb{N}$ instead of $\mathbb{Z}$ would bring back problems of initialization and saturation. We have considered and discarded several solutions to work with $\mathbb{N}$: they either complicated proofs by requiring to handle additional non-trivial cases, or used alternative notations that made theorems more difficult to read and compare to their usual signal processing formulation. For example, in order to understand some statements, one is required to know that $0-2$ is equal to 0 when working with $\mathbb{N}$ in Coq. Readability is very important to us as we want to spread formal methods among the digital processing community. Ultimately, we felt that the readability and the more intuitive subtraction offered by $\mathbb{Z}$ were worth adapting a few libraries and recursive constructions.

We define the three elementary operations on signals described in §2.1, namely addition, scaling by a real and time shift. We prove that the results are still signals (they take the value 0 for $k < 0$). The only interesting point is in the time shift by an integer K, which usually requires that $K \geq 0$: we arbitrarily choose that it returns the null signal when $K < 0$, so that the function is total.

Once we have signals, filters are simply defined as functions from signals to signals: `Definition filter := signal -> signal`. From the three elementary operations on signals defined above (addition, scaling by a real and time shifting), we define compatibility of a filter with addition, compatibility with scalar multiplication and shift invariance. Finally, we define `LTI_filter : filter -> Prop` (linear time invariant filter) as the conjunction of these properties.

As explained in §2.2, a filter can be SISO (Single-Input Single-Output) when it handles real signals as in the definition above, or MIMO (Multiple-Input Multiple-Output) with vector signals. A vector signal is defined as a function from $\mathbb{Z}$ to $\mathbb{R}^p$ that returns the null vector for every $k < 0$. A MIMO filter is then:

```
Definition MIMO_filter {N_in N_out : Z} :=
  @vect_signal N_in -> @vect_signal N_out.
```

As in textbooks and for the sake of readability, most of our theorems are dedicated to SISO filters. Nevertheless, we define the State-Space characterization of a filter, presented in §4.3, for MIMO filters, as we explicitly need this to study error propagation in §5.1.

3.2 Recursion over $\mathbb{Z}$

To define a filter given by its constant-coefficient difference equation (4), we need to build the output signal y by order-n recursion. In practice, we find it easier to use strong recursion which is more general. The standard library has a lemma `Coq.ZArith.Wf_Z.Zlt_0_rec` that shows that a construction by strong recursion is possible for positive indexes, whereas what we want is the actual function resulting from this construction. Therefore, we define `Z_strong_rec` which builds this function. It takes an initialization function `f_lt_0` that will only be used for $k < 0$, and a step function `f_rec` that computes the image of k for $k \geq 0$, depending on the image of j for any $j < k$. If we note $f|_{<k}$ the restriction of f to $(-\infty, k)$, `Z_strong_rec` builds the function $f : \mathbb{Z} \to T$ such that

$$\begin{cases} \forall k < 0, f(k) = \texttt{f\_lt\_0}(k) \\ \forall k \geq 0, f(k) = \texttt{f\_rec}(k)(f|_{<k}) \end{cases} \tag{5}$$

```
Definition Z_strong_rec (f_lt_0 : Z -> T)
                                   (f_rec : Z -> (Z -> T) -> T) (k : Z) : T := (...)
```

But, as we want to work with total functions, the second argument of `f_rec` is total. When we write $f|_{<k}$, we actually mean that it is equal to previously computed values of f for $j < k$, and to default values for $j \geq k$. As `f_rec` is just an argument of `Z_strong_rec`, it could evaluate $f|_{<k}$ for $j \geq k$, which would make no sense from a recursion perspective. To have a proper recursive construction, the argument `f_rec` needs to verify the property `f_rec_well_formed`, which means that when we call `f_rec (k : Z) (g : Z -> T)` with $k \geq 0$, the result does not depend on the values of `g` for $j \geq k$. Lemmas `Z_strong_rec_lt_0` and `Z_strong_rec_ge_0` express (5), the second one needing the hypothesis that the argument `f_rec` is well-formed. They serve as an interface so that outside of their own proofs, the 13-line definition of `Z_strong_rec` is never expanded.

```
Definition f_rec_well_formed (f_rec : Z -> (Z -> T) -> T) : Prop :=
  forall (k : Z) (g1 g2 : Z -> T), (k >= 0)%Z ->
  (forall j : Z, (j < k)%Z -> g1 j = g2 j) -> f_rec k g1 = f_rec k g2.
Lemma Z_strong_rec_ge_0 f_lt_0 f_rec k :
  (k >= 0)%Z -> f_rec_well_formed f_rec ->
    Z_strong_rec f_lt_0 f_rec k = f_rec k (Z_strong_rec f_lt_0 f_rec).
```

For example, consider a signal x defined by the recursive relation $x(k) = x(k-1) + x(k-2)$. We define the function representing this relation:
`f_rec := fun (k : Z) (g : Z -> R) => (g(k-1)%Z + g(k-2)%Z)%R`, and we can prove that it is well-formed. As a signal should be zero for $k < 0$, the initialization function is `f_lt_0 := fun (k : Z) => 0%R`. Then, `(Z_strong_rec f_lt_0 f_rec) : Z -> R` is the function that corresponds to our signal x.

3.3 Adapting existing libraries to relative indexes

We build upon the Coquelicot library to obtain sums of consecutive terms of a sequence and matrices that are compatible with relative indexes. Matrices with relative indexes look strange, but there is no problem in practice, as relevant indexes for a given matrix of size $\mathtt{h} \times \mathtt{w}$ are only subsets in any case: $1 \leq i \leq \mathtt{h}$ and $1 \leq j \leq \mathtt{w}$. We were careful to handle matrices without relying on the particular Coquelicot definition, so that we can easily switch to other existing libraries. In particular, we may use the Mathcomp library [13] and rely on its linear algebra theorems to evaluate the WCPG defined in §5.2.

4 Filter realizations

To implement a filter, one needs an explicit algorithm, which produces at each k an output $y(k)$ from the input $u(k)$ and its internal state. In the literature, a lot of algorithms exist to implement a linear filter [11]: Direct Forms, State-Space, Second-Order Sections, cascade or parallel decomposition, Lattice Wave Digital filters [14], δ- or ρ-operator based filters [15,16], etc. Each of them presents some advantages with respect to the number of coefficients, the software or hardware adequacy, the finite-precision behavior, etc. and the choice of the realization is itself a research domain [17].

We present here three classical realizations: Direct Form I (that comes from the constant-coefficients difference equation (4)), Direct Form II (that uses the same coefficients in another algorithm) and State-Space. In practice, these realizations are explained by a data-flow graph, that describes how data (signals) are processed by a filter in terms of inputs and outputs.

4.1 Direct Form I

Direct Form I (DFI) comes directly from the constant-coefficient difference equation presented in §2.4: $y(k) = \sum_{i=0}^{n} b_i u(k-i) - \sum_{i=1}^{n} a_i y(k-i)$. It depends on the $2n+1$ transfer-function coefficients $a_1, a_2, ..., a_n, b_0, b_1, ..., b_n$.

```
foreach $k$ do
    $y(k) \leftarrow \sum_{i=0}^{n} b_i u(k-i) - \sum_{i=1}^{n} a_i y(k-i)$
end
```

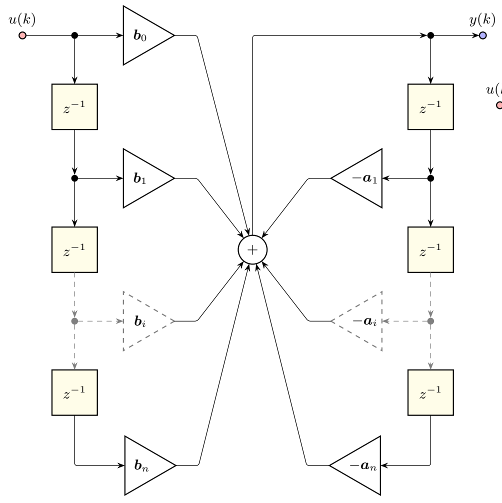
(a) Direct Form I

```
foreach $k$ do
    $e(k) \leftarrow u(k) - \sum_{i=1}^{n} a_i e(k-i)$
    $y(k) \leftarrow \sum_{i=1}^{n} b_i e(k-i)$
end
```

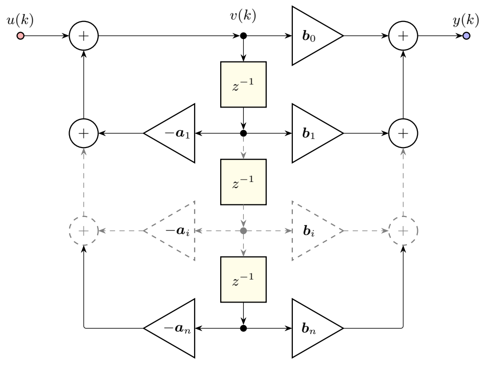
(b) Direct Form II

Fig. 2: Direct Form I and II algorithms.

The corresponding algorithm is presented in Figure 2a and the data flow graph in Figure 3a. The real program is slightly more complex: the n previous



(a) Direct Form I



(b) Direct Form II

Fig. 3: Direct Form I and II data-flow graphs.

values of u and y are stored in memory (the z^{-1} squares in the data flow graph in Figure 3a represent *delay*, each one is a memory element).

In Coq, we define a type `TFCoeffs` for the transfer function coefficients. They are given as an order n and two sequences $a, b : \mathbb{Z} \to \mathbb{R}$. In practice, n should be positive, and we will only use the values $a(i)$ for $1 \leq i \leq n$ and the values $b(i)$ for $0 \leq i \leq n$. So far, we have not needed to enforce this in the definition.

```
Record TFCoeffs := { TFC_n : Z ; TFC_a : Z -> R ; TFC_b : Z -> R }.
```

The main component needed for our recursive construction of a signal x is a function, noted `f_rec` in our definition of strong recursion (§3.2), that builds $x(k)$ for $k \geq 0$ given all previous values of x. `(DFI_f_rec tfc u) : Z -> (Z -> R) -> R` is such a function: it expresses the recursive relation characterizing the output signal for input `u` of the filter defined by Direct Form I with the coefficients `tfc`.

From this, we define the function that builds a filter from its transfer function coefficients using Direct Form I, called `filter_from_TFC` because Direct Form I is the canonical way to build a filter from these coefficients.

```
Definition DFI_f_rec (tfc : TFCoeffs) (u : signal) :=
  (fun (n :  Z) (y_before_n : Z -> R) =>
      (sum 0 (TFC_n tfc) (fun i => (TFC_b tfc i) * u (n-i)%Z)%R)
    - (sum 1 (TFC_n tfc) (fun i => (TFC_a tfc i) * y_before_n (n-i)%Z)%R)).
Definition filter_from_TFC (tfc : TFCoeffs) : filter :=
  fun (u :  signal) => build_signal_rec (DFI_f_rec tfc u).
```

Finally, we prove that a filter built this way is LTI. This proof presents no real difficulty once we have adequate lemmas about sum of consecutive value of a sequence and recursive building of a signal.

4.2 Direct Form II

Direct Form II is quite similar to Direct Form I. It uses the same coefficients, those of the transfer function. The main difference is that it only requires n delays (instead of $2n$), so it is more efficient to implement. It can be described by the data flow graph of Figure 3b, or by the algorithm of Figure 2b.

Where Direct Form I builds the output y from previous values of both y and the input u, Direct Form II builds an intermediary signal e from previous values of itself and only the current value of u, then it builds y from previous values of e. The Coq definitions reflect this. As the construction of e is recursive, we define `DFII_u2e_f_rec tfc u`, which is the main function allowing to build e given the transfer function coefficients and an input signal.

Combining the constructions of the intermediary signal e and of the output signal y, we define `filter_by_DFII` which builds a filter from transfer function coefficients using Direct Form II.

```
Definition DFII_u2e_f_rec (tfc : TFCoeffs) (u : signal) :=
  fun (n : Z) (e_before_n : Z -> R) =>
    u n - sum 1 (TFC_n tfc) (fun i => (TFC_a tfc i) * e_before_n (n-i)%Z).
Definition DFII_e2y (tfc : TFCoeffs) (e : signal) := Build_signal
  (fun n => sum 0 (TFC_n tfc) (fun i => (TFC_b tfc i) * e (n-i)%Z)%R)
  (DFII_e2y_prop tfc e).
Definition filter_by_DFII (tfc : TFCoeffs) : filter :=
  fun (u : signal) => DFII_e2y tfc (build_signal_rec (DFII_u2e_f_rec tfc u)).
```

Finally, as Direct Form I and Direct Form II use the same coefficients, which are also the coefficients associated to the transfer function, implementing either of these algorithms with the same set of coefficients should produce the same filter, meaning the same output even if the algorithms are different.

```
Theorem DFI_DFII_same_filter (tfc : TFCoeffs) :
  filter_from_TFC tfc = filter_by_DFII tfc.
```

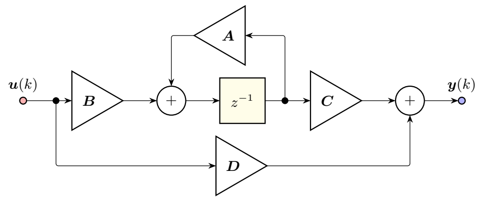
Since a filter built using `filter_from_TFC` (which is by definition built using Direct Form I) is LTI, so is a filter built using Direct Form II.

4.3 State-Space

We now consider MIMO filters, which are needed for the error analysis in §5.1. For a q-input p-output MIMO filter $\mathscr{H}$, the State-Space representation is described by four matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$. The corresponding algorithm is:

$$\begin{cases} \boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) \\ \quad\;\; \boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) \end{cases} \tag{6}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^{n\times 1}$, $\boldsymbol{u}(k) \in \mathbb{R}^{q\times 1}$ and $\boldsymbol{y}(k) \in \mathbb{R}^{p\times 1}$ are the state vector, input vector and output vector, respectively. The matrices $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times q}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$ and $\boldsymbol{D} \in \mathbb{R}^{p\times q}$ characterize the filter $\mathscr{H}$. Figure 4a exhibits its data-flow graph, and Algorithm 4b its algorithm (scalar rewriting of (6)).



(a) Data-flow graph.

```
foreach $k$ do
    for $i \in \{1, \ldots, n\}$ do
        $\boldsymbol{x}_i(k+1) \leftarrow \sum_{j=1}^{n} \boldsymbol{A}_{ij}\boldsymbol{x}_j(k) + \sum_{j=1}^{q} \boldsymbol{B}_{ij}\boldsymbol{u}_j(k)$
    end
    for $i \in \{1, \ldots, p\}$ do
        $\boldsymbol{y}_i(k) \leftarrow \sum_{j=1}^{n} \boldsymbol{C}_{ij}\boldsymbol{x}_j(k) + \sum_{j=1}^{q} \boldsymbol{D}_{ij}\boldsymbol{u}_j(k)$
    end
end
```

(b) Algorithm

Fig. 4: State-Space data-flow graph and algorithm.

In Coq, we define a State-Space as a record containing the size of the state vector and the four matrices. Here `@mtx R_Ring h w` is the type of matrices of size `h` × `w` with coefficients in the ring $\mathbb{R}$.

```
Context { N_in N_out : Z }.
Record StateSpace := { StSp_n : Z ; (* size of the state vector *)
                       StSp_A : @mtx R_Ring StSp_n StSp_n ;
                       StSp_B : @mtx R_Ring StSp_n N_in ;
                       StSp_C : @mtx R_Ring N_out StSp_n ;
                       StSp_D : @mtx R_Ring N_out N_in       }.
```

For a given State-Space and a given input vector signal u, we first define the vector signal of state vectors x by recursion, then the input vector signal y from u and x, following closely the relationship in (6). We obtain a function that builds the MIMO filter corresponding to a State-Space. We also define a SISO State-Space as a State-Space where the context variables `N_in` and `N_out` are both 1. Then we define `filter_from_StSp` that associates a SISO filter to such a State-Space. We also prove that a filter built from a State-Space is always LTI.

The impulse response h (introduced in §2.3) of a filter defined by a State-Space is proved to be computable from the matrices of the State-Space with:

$$h(k) = \begin{cases} \boldsymbol{D} & \text{if } k = 0 \\ \boldsymbol{C}\boldsymbol{A}^{k-1}\boldsymbol{B} & \text{if } k > 0 \end{cases} \tag{7}$$

Moreover, in SISO, a State-Space can also be built from the transfer function coefficients. Below is one of many ways to build such a State-Space, requiring $(n+1)^2$ coefficients for the four matrices combined (note that $\boldsymbol{C}$ is a single-row matrix, $\boldsymbol{B}$ single-column, and $\boldsymbol{D}$ single-row single-column):

$$
\begin{aligned}
\boldsymbol{A} &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots\dots & & 0 & 1 \\ -a_n & \dots\dots & & -a_2 & -a_1 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\
\boldsymbol{C} &= \begin{pmatrix} b_n - a_n b_0 & \dots\dots & b_1 - a_1 b_0 \end{pmatrix}, \qquad \boldsymbol{D} = \begin{pmatrix} b_0 \end{pmatrix}
\end{aligned}
\tag{8}
$$

```
Notation n := (TFC_n tfc).
Definition SISO_StSp_from_TFC : SISO_StateSpace := @Build_StateSpace 1 1
  (*StSp_n*) n
  (*StSp_A*) (make_mtx n n (fun i j =>
               if Z_eq_dec i n then (− (TFC_a tfc (n+1−j)))
               else if Z_eq_dec (i+1) j then 1 else 0))
  (*StSp_B*) (make_mtx n 1 (fun i j => if Z_eq_dec i n then 1 else 0))
  (*StSp_C*) (make_mtx 1 n (fun i j =>
               TFC_b tfc (n+1−j) − TFC_a tfc (n+1−j) ∗ TFC_b tfc 0))
  (*StSp_D*) (mtx_of_K (TFC_b tfc 0)).
Theorem StSp_TFC_same_filter (tfc : TFCoeffs) : (TFC_n tfc >= 0)%Z −>
  filter_from_StSp (SISO_StSp_from_TFC tfc) = filter_from_TFC tfc.
```

We define a Coq function that associates a (Single Input Single Output) State-Space to transfer function coefficients, by simply constructing the matrices as in (8). More importantly, we prove that the filter built from such a State-Space is the same as the filter obtained directly from the transfer function coefficients.

As we have seen previously, there are at least two ways to build a filter directly from these coefficients: Direct Form I and Direct Form II. Although Direct Form I is the canonical one, it is much easier to use Direct Form II in this proof. The main step is to prove that for any k, the state vector $\boldsymbol{x}(k)$ contains $e(k-n)$ as its first coefficient, $e(k-(n-1))$ as its second one, etc. up to $e(k-1)$ as its n-th coefficient, where e is the auxiliary signal that appears in Direct Form II (Figure 2b).

This is done by strong induction, with trivial initialization for $k < 0$ since everything is zero. For $k \geq 0$, the *ones* just above the diagonal in $\boldsymbol{A}$ and the *zeros* in $\boldsymbol{B}$ mean that the coefficients 1 to $n-1$ of $\boldsymbol{x}(k)$ are the coefficients 2 to n of $\boldsymbol{x}(k-1)$, so the only point left to prove is that $\boldsymbol{x}(k)_n = e(k-1)$. This is ensured by the last line of $\boldsymbol{A}$ and the last coefficient of $\boldsymbol{B}$, which make the matrix operations in (6) unfold into exactly the computation of $e(k)$ in Figure 2b.

Note that the matrices $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$ are not uniquely defined for a given filter: distinct State-Spaces describe distinct algorithms, but they may build the same filter. So it is possible to search for the *optimal* State-Space when considering the effect of the finite precision degradations [18].

5 Error analysis tools

We now aim at giving tools for a future full error analysis of implemented filters. Using finite precision arithmetic (floating- or fixed-point arithmetic), some arithmetic operations may introduce errors (mainly because the number of bits to represent the values is finite, and may be not enough to represent the exact result). These errors are then to be taken into account in the following computations as they may accumulate over time. To bound this accumulation, §5.1 shows that these errors may be extracted from the main computations. Their values are then modified over time by to another filter. To bound this error, we now only need to bound the maximal value of a filter (this may also help us prevent overflows). This is done in §5.2 by the Worst-Case Peak Gain Theorem.

5.1 Error filter

Let us focus on the errors due to finite precision arithmetic, without more details on this arithmetic. The corresponding quantization can be modeled as an extra term, called *roundoff error*. We consider a State-Space $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D})$, as this is the most general of the realizations that we have presented. Indeed, we have proven that from any other of these realizations, we can build a State-Space that defines the same algorithm and thus the same filter.

At each step, the evaluation of the states and outputs (see Algorithm 4b) is composed of sum-of-products (SoP), one per state and output. Since they are not exact, each may produce an error, compared to the exact SoP. So (6) becomes:

$$\begin{aligned} \boldsymbol{x}^*(k+1) &\leftarrow \boldsymbol{A}\boldsymbol{x}^*(k) + \boldsymbol{B}\boldsymbol{u}(k) + \boldsymbol{\varepsilon_x}(k) \\ \boldsymbol{y}^*(k) &\leftarrow \boldsymbol{C}\boldsymbol{x}^*(k) + \boldsymbol{D}\boldsymbol{u}(k) + \boldsymbol{\varepsilon_y}(k) \end{aligned} \tag{9}$$

where $\boldsymbol{x}^*(k)$ and $\boldsymbol{y}^*(k)$ are the computed values for the state vector and output vector, and $\boldsymbol{\varepsilon_x}(k)$ and $\boldsymbol{\varepsilon_y}(k)$ are the vectors of roundoff errors due to the sum-of-products evaluation. Denote $\boldsymbol{\varepsilon}(k)$ the column vector that aggregates those errors: $\boldsymbol{\varepsilon}(k) = \begin{pmatrix} \boldsymbol{\varepsilon_x}(k) \\ \boldsymbol{\varepsilon_y}(k) \end{pmatrix} \in \mathbb{R}^{n+p}$, where n is the size of $\boldsymbol{x}(k)$ and p is the size of $\boldsymbol{y}(k)$.

In order to capture the effects of the finite precision implementation we must take into account the propagation of the roundoff errors through the data-flow. The output error $\boldsymbol{\Delta y}(k)$ is defined as the difference between the outputs of the implemented and the exact filters: $\boldsymbol{\Delta y}(k) = \boldsymbol{y}^*(k) - \boldsymbol{y}(k)$. Subtracting (6) to (9), it follows that $\boldsymbol{\Delta y}(k)$ can be seen as the output of the vector signal of errors $\boldsymbol{\varepsilon}(k)$ through the filter $\mathscr{H}_\varepsilon$ defined by the State-Space $(\boldsymbol{A}, \boldsymbol{B}_\varepsilon, \boldsymbol{C}, \boldsymbol{D}_\varepsilon)$, where $\boldsymbol{B}_\varepsilon = \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{0} \end{pmatrix}$ and $\boldsymbol{D}_\varepsilon = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{I}_p \end{pmatrix}$. Equivalently (thanks to the linearity of the considered filter), the implemented filter can be seen as the sum of the exact filter $\mathscr{H}$ and the error filter $\mathscr{H}_\varepsilon$ with $\boldsymbol{\varepsilon}(k)$ as input, as shown on Figure 5.

The filter $\mathscr{H}_\varepsilon$ expresses how the errors propagate through the filter, and knowing some properties on the roundoff errors $\boldsymbol{\varepsilon}(k)$ will lead to properties on the output errors $\boldsymbol{\Delta y}(k)$, and hence on the accuracy of the implemented algorithm.

In Coq, we assume we are given a State-Space `stsp`, a vector input signal `u` and vector error signals `err_x` and `err_y`. We also assume the obviously reasonable

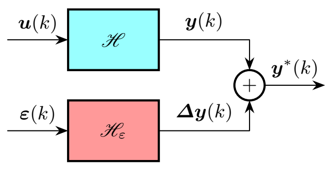

$\boldsymbol{u}(k)$
$\mathscr{H}$
$\boldsymbol{y}(k)$
$\boldsymbol{y}^*(k)$
$+$
$\boldsymbol{\varepsilon}(k)$
$\mathscr{H}_\varepsilon$
$\boldsymbol{\Delta y}(k)$

Fig. 5: Equivalent filter, with errors separated.

hypotheses that the implicit size of output vectors for `stsp` and the order of `stsp` are non-negative ((`N_out` >= 0)%`Z` and (`StSp_n stsp` >= 0)%`Z`). We define `x'` (recursively) then `y'` (using `x'`) the vector signals corresponding to $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ in (9). We define `B_err` and `D_err` the matrices corresponding to $\boldsymbol{B}_\varepsilon$ and $\boldsymbol{D}_\varepsilon$ and `errors` as the vertical concatenation of `err_x` and `err_y`. We prove that `y'` is indeed the sum of the outputs of the exact filter described by `stsp` for input `u`, and of the error filter described by a new State-Space `stsp_err` for input `errors`.

5.2 Worst-Case Peak Gain Theorem

The Worst-Case Peak Gain Theorem provides the maximum possible value for the outputs of a state-space filter. Applied on the filter $\mathscr{H}_\varepsilon$, it gives a bound on the output error due to the finite precision computations.

We consider $\mathscr{H}$ a SISO LTI filter. Its Worst-Case Peak Gain (WCPG) [19,20], noted $\langle\!\langle \mathscr{H} \rangle\!\rangle$, is an element of $\overline{\mathbb{R}}$ defined as:

$$\langle\!\langle \mathscr{H} \rangle\!\rangle = \sum_{k=0}^{\infty} |h(k)| \tag{10}$$

where h is the impulse response of $\mathscr{H}$ (see §2.3). If an input signal u is bounded by M ($\forall k,\ |u(k)| \leq M$), then the corresponding output signal y is bounded by the value $\langle\!\langle \mathscr{H} \rangle\!\rangle M \in \overline{\mathbb{R}}$. We can also write this as an inequality over $\overline{\mathbb{R}}$:

$$\forall u, \quad \sup_{k\in\mathbb{Z}}(\mathscr{H}\{u\}(k)) \ \leq\ \langle\!\langle \mathscr{H} \rangle\!\rangle \sup_{k\in\mathbb{Z}}(u(k)) \tag{11}$$

Moreover, the WCPG is optimal: it is the smallest number that verifies (11). In Coq, we define the WCPG exactly as in (10) using `Lim_seq` from Coquelicot [10].

```
Definition dirac : signal := (...) (* 0 -> 1, k <> 0 -> 0 *)
Definition impulse_response (H : filter) : signal := H dirac.
Definition sum_abs_IR (H : filter) (n : nat) :=
  sum 0 (Z.of_nat n) (fun k : Z => Rabs ((impulse_response H) k)).
Definition wcpg (H : filter) : Rbar := Lim_seq (sum_abs_IR H).
```

To prove (11), we rely on the fact that the image by a LTI filter of a signal u can be obtained as the convolution of u and the impulse response of the filter: $\mathscr{H}\{u\}(k) = \sum_{l\in\mathbb{Z}} u(l)h(k-l)$ (3). We also prove the optimality of the WCPG, with two theorems depending on whether the WCPG is finite. For both of them, from the definition of $\langle\!\langle \mathscr{H} \rangle\!\rangle$ as an infinite sum, we can get an index N such that $\sum_{k=0}^{N} |h(k)|$ is sufficiently close to $\langle\!\langle \mathscr{H} \rangle\!\rangle$. Then, we define a signal u such that for $0 \leq k \leq N$, $u(k)$ is in $\{1, -1\}$ and has the same sign as $h(N-k)$. We obtain $\mathscr{H}\{u\}(N) = \sum_{l \in \mathbb{Z}} u(l)h(N-l) = \sum_{0 \leq l \leq N} |h(l)|$.

```
Theorem wcpg_theorem (H : filter) (u : signal) (M : R) :
  LTI_filter H -> (forall k : Z, Rabs (u k) <= M) ->
    (forall k : Z, Rbar_le (Rabs (H u k)) (Rbar_mult wcpg M)).
Theorem wcpg_optimal (H : filter) : LTI_filter H -> is_finite (wcpg H) ->
  forall epsilon : R, epsilon > 0 -> exists (u : signal) (N : Z),
    ( (forall k : Z, Rabs (u k) <= 1) /\ (H u) N > wcpg H - epsilon ).
Theorem infinite_wcpg_optimal (H : filter) : LTI_filter H ->
  wcpg H = p_infty -> forall M : R, exists (u : signal) (N : Z),
    ( (forall k : Z, Rabs (u k) <= 1) /\ H u N > M ).
```

Another important property of a filter is that, for any bounded input ($\exists M$, $\forall k$, $|u(k)| \leq M$), the output is bounded as well (by a number M'). This property is known as the Bounded Input Bounded Output (BIBO) stability [21], and is shared by most filters that are of practical interest. We easily defined the `bounded` property for signals, and the `BIBO` property for filters. An important theorem is that the WCPG of a LTI filter verifying this property is always finite.

```
Theorem BIBO_wcpg_finite (H : filter) :
  LTI_filter H -> BIBO H -> is_finite (wcpg H).
```

The principle is to prove the contrapositive: if the WCPG is infinite then we will build an input u bounded by 1 such that the output is unbounded. As seen in the proof of optimality of the WCPG, for any bound M, we can construct an input u bounded by 1 and an index N such that $|\mathscr{H}\{u\}(N)| > M$. Two facts allow us to get from this construction, where u can be chosen after M, to a single unbounded signal. Firstly, the property $|\mathscr{H}\{u\}(N)| > M$ only depends on values of u for $0 \leq k \leq N$. Secondly, we are able to adapt this construction to leave an arbitrary number of preset input values unchanged: for any $M \in \mathbb{R}$, $K \in \mathbb{Z}$ and imposed values $u(0), ..., u(K)$ that are bounded by 1, we can extend them into a signal u still bounded by 1 such that $|\mathscr{H}\{u\}(N)| > M$ for some index N. Iterating this construction, we build a signal such that its image has at least a term exceeding 0, a term exceeding 1, and so on at least a term exceeding M for any $M \in \mathbb{Z}$. These repeated constructions are tricky to handle in Coq, and we used the `{...|...}` strong existential construction rather than the existential quantifier.We relied on the Limited Principle of Omniscience (LPO) to be able to construct indices N as described above.

6 Conclusions and perspectives

We have formalized digital filters in the Coq proof assistant and provided several realizations. For one of these realizations, namely the State-Space, we have proved theorems about error analysis, that will be useful when finite-precision arithmetic will come into play. A surprising difficulty was the induction on $\mathbb{Z}$ as described in §3.2: it was to decide that the standard library results were not exactly what we needed and to state the corresponding theorems and total functions.

A part of the digital processing results we did not focus on is the transfer function. We defined it but we have not yet linked it to the rest of the development. The z-Transform has been formalized in HOL [12,3] and it will be interesting to see if similar theorems may be proved with our Coq formalization.

The Worst-Case Peak Gain Theorem has been proved for the SISO filters, including State-Space. The general formula $\langle\langle \mathscr{H} \rangle\rangle = |\boldsymbol{D}| + \sum_{k=0}^{\infty} \left|\boldsymbol{C}\boldsymbol{A}^k\boldsymbol{B}\right|$ has been proved with $\boldsymbol{D}$ and $\boldsymbol{C}\boldsymbol{A}^k\boldsymbol{B}$ being 1×1 matrices, implicitly converted to real numbers. To be applied on the error filter, the proof needs to be generalized to MIMO filters, which is not difficult but cumbersome due to matrix manipulation.

To handle more realizations and develop proofs (such as error analysis proofs) only once, we may use another realization called the Specialized Implicit Framework (SIF) [22]. It was designed as a unifying tool to describe and encompass all the possible realizations of a given transfer function (like the direct forms, State-Spaces, cascade or parallel decomposition, etc.). SIF is an extension of the State-Space realization, modified to allow chained Sum-of-Products operations.

A natural perspective is to handle floating-point and fixed-point computations. Indeed, digital filters are run on embedded software that cannot compute with real numbers. As far as floating-point arithmetic is concerned, the Flocq library [23] will suit our needs, but fixed-point will be more complicated. Even if Flocq has a fixed-point format and the corresponding theorems, we want to take overflow into account and this is hardly done within Flocq: only the IEEE-754 formalization of binary floating-point numbers assumes an upper bound on the exponent. Moreover, we may want to have several handling of overflow as done in [4]. We want at least three modes: i) ensuring that no overflow happens; ii) two's complement arithmetic, where a modulo operation is used when overflow happens; iii) saturation arithmetic, where the maximal value is used when overflow happens. Adding two's complement and overflow modes to Flocq will be a necessary step towards the formal proof of the behaviors of embedded digital filters.

The final use of this work is to handle industrial applications, like the filters and controllers used in telecommunication, automotive or aeronautic with the following flow. Some filter specifications (like a transfer function) first gives an algorithm to realize the filter (like the Direct Forms or the State-Space). Then it is transformed in finite-precision code to be executed on a specific target. The bound of the output error (due to the finite-precision arithmetic) will be then deduced and proved.

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