POLE SENSITIVITY STABILITY RELATED MEASURE OF FWL REALIZATIONS WITH THE IMPLICIT STATE-SPACE FORMALISM

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Abstract: The pole-sensitivity approach is an interesting way to analyze the stability of discrete-time control system with a Finite Word Length implemented digital controller. Previous works have introduced a pole-sensitivity stability related measure in this context, and applied it to shift or δ-realizations.
This paper generalizes then by considering a more general representation, a particular implicit state-space realization, which is recently known to encompass both classical shift or δ-realization, as well as other interesting parametrizations. Finally, the problem consisting in finding realizations optimizing the FWL closed-loop stability related measure is considered.

Keywords: FWL implementation, stability related measure, pole-sensitivity

1. INTRODUCTION

Analysis of the effect of quantization and rounding in digital controllers is an important topic. The Finite Word length (FWL) implementation of digital systems or filters may alter their behavior, leading sometimes to instability. The deterioration however is not intrinsic and depends on the parameterization used during implementation. One objective is then to find the *best parameterization*, minimizing the deterioration according to pertinent indicators such as parametric sensitivity and pole-sensitivity measures (Gevers and Li, 1993; Istepanian and Whidborne, 2001).
In a previous paper (Hilaire *et al.*, 2005*b*), the authors have proposed a new framework, unifying different parameterizations possible for implementation. Using a specific implicit state-space representation, it encompasses shift, delta, observer-state feedback realizations, and many others. How to generalize the stability related measure, first proposed by Li (1998), in this new context is the aim of the present paper. The problem of its computation will be considered. Finally, the problem will consist in finding an *optimal parameterization* according to this criteria.
This paper first presents the pole-sensitivity stability related measure in the classical state-space case. Section 3 exhibits the implicit state-space realizations. Section 4 generalizes the pole-sensitivity stability related measure in this new framework, and section 5 explains how to search parametrizations that optimize this measure. Finally, section 6 presents the results obtained with different realizations on a fluid power speed control system and section 7 concludes.

[1] This work was supported by PSA Peugeot Citroën.
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2. A POLE-SENSITIVITY STABILITY RELATED MEASURE

In (Istepanian and Whidborne, 2001; Chen *et al.*, 2002; Wu *et al.*, 2000), the FWL impact of closed-loop realizations is studied. The question to be answered is : how much the digital approximation of the controller coefficients (due to FWL quantization) affects the closed-loop stability of the system.
A strictly proper discrete plant $\mathcal{P}$ is considered and (A_p, B_p, C_p) is one of its realization. The input/output equations are

$$\mathcal{P}\begin{cases} X^p_{k+1} = A_p X^p_k + B_p(R_k + Y_k) \\ U_k = C_p X^p_k \end{cases} \tag{1}$$

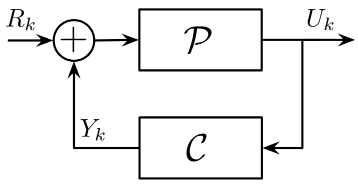

Fig. 1. Block-diagram of the closed-loop system

This plant is controlled (see figure 1) via a discrete LTI controller $\mathcal{C}$, with realization (A, B, C, D) :

$$\mathcal{C}\begin{cases} X_{k+1} = AX_k + BU_k \\ Y_k = CX_k + DU_k \end{cases} \tag{2}$$

A representation of the closed-loop system is then

$$\begin{cases} \bar{X}_{k+1} = \bar{A}\bar{X}_k + \bar{B}R_k \\ U_k = \bar{C}\bar{X}_k \end{cases} \tag{3}$$

with

$$\bar{A} \triangleq \begin{pmatrix} A_p + B_p D C_p & B_p C \\ BC_p & A \end{pmatrix} \\ \bar{B} \triangleq \begin{pmatrix} B_p \\ 0 \end{pmatrix} \bar{C} \triangleq \begin{pmatrix} C_p & 0 \end{pmatrix} \tag{4}$$

Let $X = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ represents the *realization* or the *parametrization* in the sense used in (Gevers and Li, 1993) and $\lambda_i(\bar{A}(X))_{1\leqslant i\leqslant l+n}$ be the eigenvalues of $\bar{A}(X)$.

The Pole-Sensitivity Stability related Measure (PSSM), previously proposed, evaluates how a modification ΔX of the implemented parameters X can cause a system instability : it is determined by how close the eigenvalues of $\bar{A}(X)$ are to 1 and how sensitive they are to the controller parameters perturbations. It is defined by :

$$\mu(X) \triangleq \min_{k\leqslant l+n} \frac{1 - \left|\lambda_k(\bar{A}(X))\right|}{\sqrt{N\Psi_k}} \tag{5}$$

where N is the number of non-trivial elements in X (non-zero elements in ΔX) and Ψ_k is the pole sensitivity of the closed-loop with respect to the parameters :

$$\Psi_k \triangleq \sum_{i,j} (W_X)_{i,j} \left| \frac{\partial\left|\lambda_k(\bar{A}(X))\right|}{\partial X_{i,j}} \right|^2 \tag{6}$$

$$= \left\| \frac{\partial\left|\lambda_k(\bar{A}(X))\right|}{\partial X} \times W_X \right\|_F^2 \tag{7}$$

$\times$ is the *Schur* (or *Hadamard*) product, $\|.\|_F$ the Frobenius norm and W_X is the weighting matrix associated to the realization matrix X, defined by

$$(W_X)_{i,j} = \begin{cases} 0 & \text{if } X_{i,j} \text{ is \textit{exactly} implemented} \\ 1 & \text{if not} \end{cases}$$

It allows to ignore the coefficients that do not contribute to the FWL deterioration (these coefficients are those that are exactly implemented, like 0, ± 1 or power of 2). This is also a sparseness consideration because the computational effort could be less with those coefficients.

This measure could be directly linked to an estimation of the smallest word-length needed to implement the controller that can guarantee the closed-loop stability in a fixed-point or floating-point processor (see (Wu *et al.*, 2003)).

3. IMPLICIT STATE-SPACE FRAMEWORK

(Hilaire *et al.*, 2005*b*) highlights the interest of the implicit state-space representation in the context of FWL implementation problems and proposes to use a specialized form directly connected to the in-line computations to be performed. It can be used as a unifying framework to allow a more detailled (still macroscopic) description of FWL implementations. Various structures of realizations, like q or δ-realizations, Observer-State-Feedback forms, classical forms like Direct Form I and II (and also mixed structures) may be then described in a single unifying form.
Equation (8) recalls the specialized implicit form proposed that makes explicit the parametrization and the intermediate variables used.

$$\begin{pmatrix} J & 0 & 0 \\ -K & I & 0 \\ -L & 0 & I \end{pmatrix} \begin{pmatrix} T_{k+1} \\ X_{k+1} \\ Y_k \end{pmatrix} = \begin{pmatrix} 0 & M & N \\ 0 & P & Q \\ 0 & R & S \end{pmatrix} \begin{pmatrix} T_k \\ X_k \\ U_k \end{pmatrix} \tag{8}$$

where

- the matrix J is lower triangular with 1 on the diagonal
- T_{k+1} is the intermediate variable in the calculations of step k (the column of 0 in the second matrix shows that T_k is not used for the calculation at step k : that characterizes the concept of intermediate variable)
- X_{k+1} is the stored state-vector (X_k is effectively stored from one step to the next, in order to compute X_{k+1} at step k)

T_{k+1} and X_{k+1} form the state-vector : X_{k+1} is stored from one step to the next, while T_{k+1} is computed and used inside one time step.

It is implicitly considered throught the paper that the computations associated to the realization (8) are ordered from top to bottom, associated in a one to one manner to the following algorithm :

[1] $J.T_{k+1} \leftarrow M.X_k + N.U_k$:
calculation of the intermediate variables. J is lower triangular, so $T_{k+1}^{(0)}$ is first calculated, and then $T_{k+1}^{(1)}$ using $T_{k+1}^{(0)}$ and so on ... (There's no need to compute J^{-1})
[2] $X_{k+1} \leftarrow K.T_{k+1} + P.X_k + Q.U_k$
[3] $Y_k \leftarrow L.T_{k+1} + R.X_k + S.U_k$

(Steps [2] and [3] can be changed around : the computational delay could be reduced by evaluating Y_k first).
J beeing nonsingular, equation (8) is equivalent in infinite precision to the classical state-space form

$$\begin{pmatrix} T_{k+1} \\ X_{k+1} \\ \hline Y_k \end{pmatrix} = \left(\begin{array}{cc|c} 0 & J^{-1}M & J^{-1}N \\ 0 & A & B \\ \hline 0 & C & D \end{array}\right) \begin{pmatrix} T_k \\ X_k \\ \hline U_k \end{pmatrix} \quad (9)$$

with

$$A = KJ^{-1}M + P \quad (10)$$
$$B = KJ^{-1}N + Q \quad (11)$$
$$C = LJ^{-1}M + R \quad (12)$$
$$D = LJ^{-1}N + S \quad (13)$$

However, (9) corresponds to a different parametrization than the one in (8).
The transfer function considered may be then defined by

$$H(z) = C(zI_n - A)^{-1}B + D \quad (14)$$

In the following, a realization will be defined in the implicit form by its parameters used for the internal description

$$\mathcal{R} :\triangleq (J, K, L, M, N, P, Q, R, S) \quad (15)$$

It could also be equivalently written in a compact form with parameter Z, with

$$Z \triangleq \begin{pmatrix} -J & M & N \\ K & P & Q \\ L & R & S \end{pmatrix} \quad (16)$$

A "*structuration*" will be a subset of realizations with a special structure : some coefficients or some dimensions of the realization matrices are then *a priori* fixed. For example, a δ-realization can be written using the shift-operator with the implicit proposed state-space form (see (Hilaire *et al.*, 2005*b*)) :

$$\begin{pmatrix} I & 0 & 0 \\ -\Delta I & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} T_{k+1} \\ X_{k+1} \\ Y_k \end{pmatrix} = \begin{pmatrix} 0 & A_\delta & B_\delta \\ 0 & I & 0 \\ 0 & C_\delta & D_\delta \end{pmatrix} \begin{pmatrix} T_k \\ X_k \\ U_k \end{pmatrix} \quad (17)$$

So a δ-structuration is the subset of realizations $\mathcal{R}$ where

$$\mathcal{R} := (I, \Delta I, 0, A_\delta, B_\delta, I, 0, C_\delta, D_\delta) \quad (18)$$

4. STABILITY RELATED MEASURE APPLIED TO IMPLICIT STATE-SPACE FRAMEWORK

If the numerical FWL description of the controller $\mathcal{C}$ is written within the Implicit State-Space framework (8), we can extend the pole-sensitivity stability related measure, and then evaluate the measure according to different realizations.

Let Z be the matrix of an implicit state-space realization ($\bar{A}$ depends on Z through equations (4) and (10) to (13)). Let also $(\lambda_k)_{1 \leqslant k \leqslant l+n}$ denote the set of the $l+n$ eigenvalues of $\bar{A}(Z)$.

In this new context, the PSSM may be written as

$$\mu(Z) = \min_{k \leqslant l+n} \frac{1 - |\lambda_k|}{\sqrt{N \Psi_k}} \quad (19)$$

where N represents again the number of non trivial elements ($N = \|W_Z\|_F^2$) and

$$\Psi_k = \left\| \frac{\partial |\lambda_k|}{\partial Z} \times W_Z \right\|_F^2 \quad (20)$$

The computation of $\mu(Z)$ requires the evaluation of $\frac{\partial |\lambda_k|}{\partial Z}$. This can be done thanks to the original result of proposition 1 below.

Lemma 1. (Li, 1998; Istepanian and Whidborne, 2001)
Let us consider a differentiable function $f : \mathbb{R}^{m \times n} \to \mathbb{C}$, and two matrices $M \in \mathbb{R}^{m \times n}$ and $X \in \mathbb{R}^{p \times q}$. Let M_0, M_1 and M_2 be constant matrices with appropriate dimensions, then the following results hold

- *if $M = M_0 + M_1 X M_2$, then*
$$\frac{\partial f(M)}{\partial X} = M_1^\top \frac{\partial f(M)}{\partial M} M_2^\top$$
- *if $M = M_0 + M_1 X^{-1} M_2$, then*
$$\frac{\partial f(M)}{\partial X} = -\left(M_1 X^{-1}\right)^\top \frac{\partial f(M)}{\partial M} \left(X^{-1} M_2\right)^\top$$

Proposition 1. Due to the relation between J, K, L, M, N, P, Q, R, S and λ_k, the sensitivity of $|\lambda_k|$ with respect to Z is given by

$$\frac{\partial |\lambda_k|}{\partial Z} = \bar{M}_1^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \bar{M}_2^\top \quad (21)$$

with

$$\bar{M}_1 \triangleq \begin{pmatrix} B_p L J^{-1} & 0 & B_p \\ K J^{-1} & I_n & 0 \end{pmatrix} \quad (22)$$

$$\bar{M}_2 \triangleq \begin{pmatrix} J^{-1} N C_p & J^{-1} M \\ 0 & I_n \\ C_p & 0 \end{pmatrix} \quad (23)$$

Proof:
• let $Y \in \{K, L, M, N, P, Q, R, S\}$, it is clear that $\bar{A}$ depends linearly on Y ($\bar{A} = M_0 + M_1 Y M_2$). The lemma 1 leads then to :

$$\frac{\partial |\lambda_k|}{\partial K} = \begin{pmatrix} 0 \\ I \end{pmatrix}^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \left(J^{-1}NC_p \;\; J^{-1}M\right)^\top$$

$$\frac{\partial |\lambda_k|}{\partial L} = \begin{pmatrix} B_p \\ 0 \end{pmatrix}^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \left(J^{-1}NC_p \;\; J^{-1}M\right)^\top$$

$$\frac{\partial |\lambda_k|}{\partial M} = \begin{pmatrix} B_pLJ^{-1} \\ KJ^{-1} \end{pmatrix}^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \left(0 \;\; I_n\right)^\top$$

$$\frac{\partial |\lambda_k|}{\partial N} = \begin{pmatrix} B_pLJ^{-1} \\ KJ^{-1} \end{pmatrix}^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \left(C_p \;\; 0\right)^\top$$

$$\frac{\partial |\lambda_k|}{\partial P} = \begin{pmatrix} 0 \\ I \end{pmatrix}^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \left(0 \;\; I_n\right)^\top$$

$$\frac{\partial |\lambda_k|}{\partial Q} = \begin{pmatrix} 0 \\ I \end{pmatrix}^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \left(C_p \;\; 0\right)^\top$$

$$\frac{\partial |\lambda_k|}{\partial R} = \begin{pmatrix} B_p \\ 0 \end{pmatrix}^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \left(0 \;\; I_n\right)^\top$$

$$\frac{\partial |\lambda_k|}{\partial S} = \begin{pmatrix} B_p \\ 0 \end{pmatrix}^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \left(C_p \;\; 0\right)^\top$$

• $\bar{A}$ depends linearly on J^{-1}, then :

$$\frac{\partial |\lambda_k|}{\partial J} = -\begin{pmatrix} B_pLJ^{-1} \\ KJ^{-1} \end{pmatrix}^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \left(J^{-1}NC_p \;\; J^{-1}M\right)^\top$$

These equations can be written in the compact form (21) with

$$\frac{\partial}{\partial Z} = \begin{pmatrix} -\frac{\partial}{\partial J} & \frac{\partial}{\partial M} & \frac{\partial}{\partial N} \\ \frac{\partial}{\partial K} & \frac{\partial}{\partial P} & \frac{\partial}{\partial Q} \\ \frac{\partial}{\partial L} & \frac{\partial}{\partial R} & \frac{\partial}{\partial S} \end{pmatrix}$$

■

The practical computation of $\frac{\partial |\lambda_k|}{\partial \bar{A}}$ will be done according to proposition 2 :

Proposition 2. (Gevers and Li, 1993; Wu *et al.*, 2003) *Let $M \in \mathbb{R}^{n \times n}$ be diagonalisable. Let $\{\lambda_k\}$ be its eigenvalues, and $\{x_k\}$ the corresponding right eigenvectors. Denote $M_x \triangleq \left(x_1 x_2 \dots x_n\right)$ and $M_y = \left(y_1 y_2 \dots y_n\right) \triangleq M_x^{-H}$. Then*

$$\frac{\partial \lambda_k}{\partial M} = y_k^* x_k^\top \quad \forall k = 1, \dots, n \tag{24}$$

and

$$\frac{\partial |\lambda_k|}{\partial M} = \frac{1}{|\lambda_k|} Re\left(\lambda_k^* \frac{\partial \lambda_k}{\partial M}\right) \tag{25}$$

where $.^*$ denotes the conjugate operation, $Re(.)$ the real part and $.^H$ the transpose conjugate operator.

Equations (21), (24) and (25) allow the evaluation of Ψ_k and $\mu(Z)$ of any realization written in the implicit state-space formalism.
Previous results on the PSSM with shift or δ-realizations (Wu *et al.*, 2000) can be found again.
Denote $\frac{\partial |\lambda_k|}{\partial \bar{A}} = \begin{pmatrix} \alpha_k & \beta_k \\ \theta_k & \lambda_k \end{pmatrix}$, with a partition corresponding to the block partitioned structure of $\bar{A}$ in (4).
In the shift realization case :

$$\Psi_k = \|\lambda_k\|_F^2 + \left\|\theta_k C_p^\top\right\|_F^2 + \left\|B_p^\top \beta_k\right\|_F^2 + \left\|B_p \alpha_k C_p^\top\right\|_F^2$$

and in the δ-realization case :

$$\Psi_k = \Delta^2 \|\lambda_k\|_F^2 + \Delta^2 \left\|\theta_k C_p^\top\right\|_F^2 + \left\|B_p^\top \beta_k\right\|_F^2 + \left\|B_p \alpha_k C_p^\top\right\|_F^2$$

5. OPTIMAL REALIZATION

Then, since the PSSM could be measured for various equivalent realizations, it could be interesting to find realizations with a maximum tolerance to FWL stability related perturbation.
Let $\mathcal{R}_H$ denote the set of equivalent realizations of the transfer function H. The optimal design problem according to the FWL stability related measure consists in finding $\mathcal{R}^{opt}$ such that :

$$\mathcal{R}^{opt} = \underset{\mathcal{R} \in \mathcal{R}_H}{arg\ max}\ \ \mu(\mathcal{R}) \tag{26}$$

It is also possible to consider only a subspace of it, by considering specific structured realizations (a structured realization is a realization organized according to a structuration $\mathcal{S}$). The optimal structured realization design problem according to the PSSM will then consists in finding

$$\mathcal{R}^{opt} = \underset{\mathcal{R} \in \mathcal{R}_H^{\mathcal{S}}}{arg\ max}\ \ \mu(\mathcal{R}) \tag{27}$$

Some structured realizations set $\mathcal{R}_H^{\mathcal{S}}$ could be defined as the set of realization obtained from a similarity transform of an initial realization $\mathcal{R}_0 := (Z_0)$ (see (Hilaire *et al.*, 2005*a*)) : $\mathcal{R}_H^{\mathcal{S}}$ is the set of realizations $\mathcal{R} := (Z)$ such that

$$Z = \mathcal{T}_1 Z_0 \mathcal{T}_2 \tag{28}$$

with

$$\mathcal{T}_1 = \begin{pmatrix} U & & \\ & T^{-1} & \\ & & I_p \end{pmatrix}, \mathcal{T}_2 = \begin{pmatrix} V & & \\ & T & \\ & & I_m \end{pmatrix} \tag{29}$$

The closed-loop matrix for the realization $\mathcal{R} := (Z)$ can be expressed from the closed-loop matrix for the initial realization $\mathcal{R}_0 := (Z_0)$ as :

$$\bar{A}(Z) = \begin{pmatrix} I_l & \\ & T \end{pmatrix}^{-1} \bar{A}(Z_0) \begin{pmatrix} I_l & \\ & T \end{pmatrix} \tag{30}$$

So $\bar{A}(Z)$ and $\bar{A}(Z_0)$ are similar and share the same eigenvalues, independently of the similarity transformation on Z_0.

Proposition 3. The FWL stability related measure of each realization $\mathcal{R} := (Z)$ of $\mathcal{R}_H$ can be computed from the measure of the initial realization $\mathcal{R}_0 := (Z_0)$ thanks to :

$$\left.\frac{\partial |\lambda_k|}{\partial Z}\right|_Z = \mathcal{T}_1^{-\top} \left.\frac{\partial |\lambda_k|}{\partial Z}\right|_{Z_0} \mathcal{T}_2^{-\top} \tag{31}$$

Proof:
From (30), it comes :

$$\left.\frac{\partial |\lambda_k|}{\partial \bar{A}}\right|_Z = \begin{pmatrix} I_l & \\ & T \end{pmatrix}^\top \left.\frac{\partial |\lambda_k|}{\partial \bar{A}}\right|_{Z_0} \begin{pmatrix} I_l & \\ & T \end{pmatrix}^{-\top}$$

This equation is injected in (21) and terms around $\frac{\partial |\lambda_k|}{\partial \bar{A}}$ are regrouped and written with $\mathcal{T}_1^{-\top}$ and $\mathcal{T}_2^{-\top}$. ■

In the following, the controller structure is fully parameterized, which is most often the case. So the weighting matrix W_Z depends only on the structure considered ($W_Z = W_{Z_0}$).
The optimal design problem then consists in finding the non-singular matrices $\mathcal{T}_1$ and $\mathcal{T}_2$ (or T, U and V) maximizing $\mu(\mathcal{T}_1, \mathcal{T}_2)$:

$$\mu(\mathcal{T}_1, \mathcal{T}_2) = \min_k \frac{1 - |\lambda_k|}{\sqrt{N\Psi_k(\mathcal{T}_1, \mathcal{T}_2)}} \tag{32}$$

with

$$\Psi_k = \left\| \left(\mathcal{T}_1^{-\top} \left.\frac{\partial |\lambda_k|}{\partial Z}\right|_{Z_0} \mathcal{T}_2^{-\top} \right) \times W_{Z_0} \right\|_F^2$$

6. EXAMPLES

The numerical example, used here to evaluate the PSSM under various parametrizations, is a single-input single-output fluid power control system studied in (Njabeleke *et al.*, 1997). The discrete-time (sampled at 2 kHz) plant $\mathcal{P}$ is given by (A_p, B_p, C_p) in (35). The initial realization $\mathcal{R}_0 := (Z_0)$ of the controller $\mathcal{C}$ is given in controllable canonical form in equation (36). It is important to notice that the coefficients are given with only 4 digits, but, due to the sensitivity of this example, this could be not sufficient to define correctly the system. Bold font is used to exhibit parameters that risk to be approximated during the quantization process toward implementation; the weighting matrix is built accordingly.
As in (Chen *et al.*, 1999), the Adaptive Simulated Annealing algorithm (Ingber, 1996) is used here to solve the optimal structured realization design problem (27).
First, the optimal q-realization is searched with

$$\mathcal{T}_1 = \begin{pmatrix} I_q & & \\ & T^{-1} & \\ & & I_p \end{pmatrix}, \mathcal{T}_2 = \begin{pmatrix} I_q & & \\ & T & \\ & & I_m \end{pmatrix} \tag{33}$$

and fortunately the results are similar to those found in (Wu *et al.*, 2003).

Then, the optimal δ-realization, corresponding to equation (17) is searched (with $\Delta = 2^{-5}$, considered as exactly implemented) with

$$\mathcal{T}_1 = \begin{pmatrix} T^{-1} & & \\ & T^{-1} & \\ & & I_p \end{pmatrix}, \mathcal{T}_2 = \begin{pmatrix} T & & \\ & T & \\ & & I_m \end{pmatrix} \tag{34}$$

and exhibits a better FWL stability related measure (see Z_δ^{opt} in equation (37)).

A last example of structuration is here considered : the controller is split in two cascaded lower-order controller, each one is implemented with classical state-space realization (the input of the first controller is computed first to be used then as input for the second controller). This cascade realization, expressed with the Implicit State-Space framework in equation (38) where (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are the system matrices of the two subsystems, is detailed in (Hilaire *et al.*, 2005*b*). The realization exhibited in equation (39) is not an optimal realization : the choice of the split is not optimal, only each sub-realization is optimized separately, but it provides a good PSSM value.

The numerical results obtained according to the development of the paper are summarized in the following table :

realization	$\mu(\mathcal{R})$	**parameters**
canonical form q	4.4196e-12	9
optimal q	6.8714e-5	25
canonical form δ	1.1699e-5	9
optimal δ	1.7413e-3	25
cascade	1.0484e-4	18

The results obtained are coherent with existing ones : the digital controllers described with the δ-operator has better closed-loop stability robustness to FWL effects (but required a little bit more computations). The δ-canonical form could be a good compromise between FWL performance and computational efforts. The cascade realization is interesting in the sense that it shows that other structurations can provide good FWL closed-loop stability performance. The search of optimal realization (under the PSSM criterion) have to be extended to larger set of equivalent realizations, and the specialized Implicit State-Space framework provides a sufficient one, directly linked to the embedded algorithm.

7. CONCLUSION

The problem of FWL impact of controllers' implementation with closed-loop aspect has been considered in this paper. A new FWL closed-loop stability related measure, in the Implicit State-Space formalism context, has tractably been derived, which takes into account a larger set of realizations. A classical example demonstrates the proposed design procedure on various implementation schemes. Further interesting structurations (like cascaded shift/delta realization, etc...) will be examined in future works.

8. ACKNOWLEDGEMENT

The authors wish to thank J.F. Whidborne for the numerical data of the example, and PSA Peugeot Citroën for their interest and financial support.

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$$A_p = \begin{pmatrix} 9.9988e{-1} & 1.9432e{-5} & 5.9320e{-5} & -6.2286e{-5} \\ -4.9631e{-7} & 2.3577e{-2} & 2.3709e{-5} & 2.3672e{-5} \\ -1.5151e{-3} & 2.3709e{-2} & 2.3751e{-5} & 2.3898e{-5} \\ 1.5908e{-3} & 2.3672e{-2} & 2.3898e{-5} & 2.3667e{-5} \end{pmatrix} B_p = \begin{pmatrix} 3.0504e{-3} \\ -1.2373e{-2} \\ -1.2375e{-2} \\ -8.8703e{-2} \end{pmatrix} C_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^\top \tag{35}$$

$$Z_0 = \left(\begin{array}{c|cccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{-3.3071e{-1}} & 1 \\ 0 & 1 & 0 & 0 & \mathbf{1.9869e{+0}} & 0 \\ 0 & 0 & 1 & 0 & \mathbf{-3.9816e{+0}} & 0 \\ 0 & 0 & 0 & 1 & \mathbf{3.3255ee{+0}} & 0 \\ \hline 0 & \mathbf{-1.6112e{-3}} & \mathbf{-1.5998e{-3}} & \mathbf{-1.5885e{-3}} & \mathbf{-1.5773e{-3}} & \mathbf{-8.0843e{-4}} \end{array}\right) \tag{36}$$

$$Z_\delta^{opt} = \left(\begin{array}{cccc|cccc|c} 1 & 0 & 0 & 0 & \mathbf{-4.3728e{+0}} & \mathbf{2.7770e{+0}} & \mathbf{1.5953e{+1}} & \mathbf{2.1160e{+1}} & \mathbf{3.5644e{-2}} \\ 0 & 1 & 0 & 0 & \mathbf{2.3090e{+0}} & \mathbf{-1.2959e{+0}} & \mathbf{-6.6800e{+0}} & \mathbf{-9.5796e{+0}} & \mathbf{-2.6145e{-2}} \\ 0 & 0 & 1 & 0 & \mathbf{6.4736e{+0}} & \mathbf{-4.1528e{+0}} & \mathbf{-2.4059e{+1}} & \mathbf{-3.2103e{+1}} & \mathbf{-1.0745e{-2}} \\ 0 & 0 & 0 & 1 & \mathbf{-1.7320e{+0}} & \mathbf{1.0786e{+0}} & \mathbf{6.0998e{+0}} & \mathbf{8.1425e{+0}} & \mathbf{1.8563e{-2}} \\ \hline -\Delta & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\Delta & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\Delta & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\Delta & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & \mathbf{-2.8733e{+0}} & \mathbf{5.6735e{-1}} & \mathbf{-1.3643e{+0}} & \mathbf{2.7498e{+0}} & \mathbf{-8.0843e{-4}} \end{array}\right) \tag{37}$$

$$\begin{pmatrix} I & 0 & 0 \\ \begin{pmatrix} 0 \\ -B_2 \end{pmatrix} & I & 0 \\ -D_2 & 0 & I \end{pmatrix} \begin{pmatrix} T_{k+1} \\ \begin{pmatrix} X^{(1)}_{k+1} \\ X^{(2)}_{k+1} \end{pmatrix} \\ Y^{(2)}_k \end{pmatrix} = \begin{pmatrix} 0 & \begin{pmatrix} C_1 & 0 \end{pmatrix} & D_1 \\ 0 & \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} & \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & C_2 \end{pmatrix} & 0 \end{pmatrix} \begin{pmatrix} T_k \\ \begin{pmatrix} X^{(1)}_k \\ X^{(2)}_k \end{pmatrix} \\ U^{(1)}_k \end{pmatrix} \tag{38}$$

$$Z_c = \left(\begin{array}{c|cccc|c} 1 & \mathbf{2.1871e{+2}} & \mathbf{-3.5349e{+1}} & 0 & 0 & 1 \\ \hline 0 & \mathbf{9.9165e{-1}} & \mathbf{9.6165e{-4}} & 0 & 0 & \mathbf{9.4399e{-3}} \\ 0 & \mathbf{-8.3050e{-3}} & \mathbf{1.0009e{+0}} & 0 & 0 & \mathbf{2.0294e{-3}} \\ \mathbf{1.3626e{-4}} & 0 & 0 & \mathbf{7.6963e{-1}} & \mathbf{-3.1670e{-1}} & 0 \\ \mathbf{-2.8207e{-5}} & 0 & 0 & \mathbf{-3.1670e{-1}} & \mathbf{5.6321e{-1}} & 0 \\ \hline \mathbf{-8.0843e{-4}} & 0 & 0 & \mathbf{-1.4577e{-5}} & \mathbf{1.5116e{-5}} & 0 \end{array}\right) \tag{39}$$