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LOW PARAMETRIC SENSITIVITY REALIZATION DESIGN FOR FWL IMPLEMENTATION OF MIMO CONTROLLERS : THEORY AND APPLICATION TO THE ACTIVE CONTROL OF VEHICLE LONGITUDINAL OSCILLATIONS

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Abstract: The implementation of a controller in a Finite Word Length (FWL) context may lead to a deterioration of the global performance, due to parametric errors (quantification of coefficients) and numerical noises (roundoff noises). This deterioration depends on the choise of the realization used to numerically implement the controller. In previous papers, the authors have introduced a new representation allowing to unify different realizations including among others those using q or δ-operators. In this paper, the parametric sensitivity measure is generalized in the MIMO case and used with a specific realization : the Observer-State-Feedback realization. Such a realization is not unique and one problem consists in finding an optimal realization according to that measure. Looking for such a structure is applied on a pratical example : the active control of vehicle longitudinal oscillations.

Keywords: implementation, observers, embedded control algorithms, digital control

1. INTRODUCTION

The digital implementation of controllers or filters in a numerical processor is not an obvious task, specially in the Finite Word Length (FWL) case. Since the processor cannot compute with an infinite precision, the implementation leads to a degradation of the input/output relationship. This deterioration has two separate origins, the roundoff errors in the numerical computations and the quantization of the coefficients involved, and can be formalized in numerical noises and parametric errors.

Since it exists various equivalent numerical realizations of a controller or a filter, and since these realizations are no more equivalent in finite precision, the problem of FWL implementation consists in finding optimal ones with regards to deterioration criteria. Usually, some classical realizations, like with q or δ-operator, direct form I and II, or observer state feedback forms, are studied. In a previous paper (Hilaire *et al.*, 2005*b*), the authors have proposed a new framework, unifying different parametrizations possible for implementation. With a specific implicit state-space representation, it encompasses usual realizations

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and many others unexplored.
The different analysis tools may be used to determine how the realization will be altered during the FWL process : the norm of the parametric sensitivity (Gevers and Li, 1993), the pole sensitivity stability related measure (Chen *et al.*, 2002), the amount of computations, ... Those measures are based on how sensitive the transfer function, or some eigenvalues, are to the quantization process. This paper first exhibits the way to use an implicit state-space realization (first presented in (Hilaire *et al.*, 2005*b*)) in order to encapsulate, in a single framework directly related to implementation, various parametrizations usually studied separately. Section 3 generalizes the use of the parametric sensitivity measure to this implicit description in the MIMO case, while Section 4 considers the optimal design problem, according to this measures. Finally section 5 presents the results obtained on different observer-state-feedback forms, applied to a real system before concluding in section 6.

2. IMPLICIT STATE-SPACE FRAMEWORK

Hilaire *et al.* (2005*b*) highlights the interest of the implicit state-space representation in the context of FWL implementation problems and proposes to use a specialized form still macroscopic but more directly connected to the in-line calculations to be performed. Equation (1) recalls this form to make explicit the parametrization and the intermediate variables used.

$$\begin{pmatrix} J & 0 & 0 \\ -K & I & 0 \\ -L & 0 & I \end{pmatrix}\begin{pmatrix} T_{k+1} \\ X_{k+1} \\ Y_k \end{pmatrix} = \begin{pmatrix} 0 & M & N \\ 0 & P & Q \\ 0 & R & S \end{pmatrix}\begin{pmatrix} T_k \\ X_k \\ U_k \end{pmatrix} \quad (1)$$

where

- the matrix J is lower triangular with 1 on the diagonal
- T_{k+1} is the intermediate variable in the calculations of step k (the column of 0 in the second matrix shows that T_k is not used for the calculation at step k : that characterizes the concept of intermediate variables)
- X_{k+1} is the stored state-vector (X_k is effectively stored from one step to the next, in order to compute X_{k+1} at step k)

T_{k+1} and X_{k+1} form the state-vector : X_{k+1} is stored from one step to the next, while T_{k+1} is computed and used inside one time step.

It is implicitly considered throught the paper that the computations associated to the realization (1) are ordered from top to bottom. So the following algorithm is associated in a one to one manner to (1) :

[1] $J.T_{k+1} = M.X_k + N.U_k$:
calculation of the intermediate variables. J is lower triangular, so $T_{k+1}^{(0)}$ is first calculated, and then $T_{k+1}^{(1)}$ using $T_{k+1}^{(0)}$ and so on ... (There's no need to compute J^{-1})
[2] $X_{k+1} = K.T_{k+1} + P.X_k + Q.U_k$
[3] $Y_k = L.T_{k+1} + R.X_k + S.U_k$

J is nonsingular, so equation (1) is equivalent in infinite precision to the classical state-space form

$$\left(\begin{array}{c} T_{k+1} \\ X_{k+1} \\ \hline Y_k \end{array}\right) = \left(\begin{array}{cc|c} 0 & J^{-1}M & J^{-1}N \\ 0 & A & B \\ \hline 0 & C & D \end{array}\right)\left(\begin{array}{c} T_k \\ X_k \\ \hline U_k \end{array}\right) \quad (2)$$

$$A = KJ^{-1}M + P \quad (3)$$
$$B = KJ^{-1}N + Q \quad (4)$$
$$C = LJ^{-1}M + R \quad (5)$$
$$D = LJ^{-1}N + S \quad (6)$$

However, (2) corresponds to a different parametrization than the one in (1).

The transfer function considered may be then defined by

$$H(z) = C(zI_n - A)^{-1}B + D \quad (7)$$

In the following, a realization will be defined in the implicit form by its parameters used for the internal description. It can be written in a compact form with parameter Z, where $Z \in \mathbb{R}^{k\times l}$ is defined by

$$Z \triangleq \begin{pmatrix} -J & M & N \\ K & P & Q \\ L & R & S \end{pmatrix} \quad (8)$$

A structuration will be a subset of realizations with a special structure : some coefficients or some dimensions of the realization matrices are then fixed *a priori*. For example, a δ-realization can be written using the shift-operator with the implicit proposed state-space form (see (Hilaire *et al.*, 2005*b*)) :

$$\begin{pmatrix} I & 0 & 0 \\ -\Delta I & I & 0 \\ 0 & 0 & I \end{pmatrix}\begin{pmatrix} T_{k+1} \\ X_{k+1} \\ Y_k \end{pmatrix} = \begin{pmatrix} 0 & A_\delta & B_\delta \\ 0 & I & 0 \\ 0 & C_\delta & D_\delta \end{pmatrix}\begin{pmatrix} T_k \\ X_k \\ U_k \end{pmatrix} \quad (9)$$

The similarity transform

$$Z = \mathcal{T}_1 Z_0 \mathcal{T}_2 \quad (10)$$

with

$$\mathcal{T}_1 = \begin{pmatrix} U & & \\ & T^{-1} & \\ & & I_p \end{pmatrix}, \mathcal{T}_2 = \begin{pmatrix} V & & \\ & T & \\ & & I_m \end{pmatrix} \quad (11)$$

allows, in the general case, to explore a set of equivalent structured realizations.

3. THE INPUT/OUTPUT SENSITIVITY MEASURE IN THE MIMO CASE

In (Hilaire *et al.*, 2005*a*), the sensitivity measure, derived from the Gevers and Li (1993) L_2-measure, is defined for the SISO case and applied to the implicit state-space formalism :

$$M_{L_2}^W \triangleq \sum_{X\in\{J,K,L,M,N,P,Q,R,S\}} \left\| \frac{\partial \tilde{H}(z)}{\partial X} \times W_X \right\|_2^2 \quad (12)$$

where

- $\times$ denotes the *Schur* product
- W_X are the weighting matrices associated with the realization matrices (J,K,L,M, N,P,Q,R,S). They allow to only take into account the coefficients that they will have to be quantized during the implementation process (Li, 1998), and do not take in consideration the sensitivity of exactly implemented coefficients (like 0, ± 1 or power of 2). They are defined by

$$(W_X)_{i,j} = \begin{cases} 0 & \text{if } X_{i,j} \text{ is \textit{exactly} implemented} \\ 1 & \text{if not} \end{cases}$$

- $\tilde{H}(z) \triangleq H(z) - D = C(zI-A)^{-1}B$
  It is preferable to consider $\frac{\partial \tilde{H}(z)}{\partial X}$ instead of $\frac{\partial H(z)}{\partial X}$, because $\frac{\partial \tilde{H}(z)}{\partial X} = \frac{\partial H(z)}{\partial X} - \frac{\partial D}{\partial X}$ is strictly proper whatever X and have always a L_2-norm (it is strictly equivalent to compute $\frac{\partial H(z)}{\partial X}$ and do not consider the constant term in the evaluation of the L_2-norm). Moreover, $\frac{\partial D}{\partial X}$ is independent of the state-space coordinates and have not to be considered here.

One way to expand the overall L_2-sensitivity measure for the implicit state-space in the MIMO case is to consider each sub-transfer function and apply equation (12) to each of them, and then use the L_2-norm property (the square of the norm is the sum of the square of each sub-terms). Another way consists in considering the global impact of each coefficient on the transfer matrix $H(z)$ and use the definition of the derivative of a matrix with respect to another matrix.

Let's denote $\frac{\delta \tilde{H}}{\delta X}$ the matrix of the L_2-norm of the sensitivity of the transfer function $H(z)$ with respect to each coefficients $X_{i,j}$:

$$\left(\frac{\delta \tilde{H}}{\delta X}\right)_{i,j} \triangleq \left\| \frac{\partial \tilde{H}}{\partial X_{i,j}} \right\|_2 \quad (13)$$

It allows to evaluate the overall impact of each coefficient, and only take into account coefficients we need to.
Due to the L_2-norm property, we have

$$\left\| \frac{\delta \tilde{H}}{\delta X} \right\|_F = \left\| \frac{\partial \tilde{H}}{\partial X} \right\|_2 \quad (14)$$

where $\|.\|_F$ is the Frobenius norm.
So the sensitivity function in MIMO could be expressed as :

$$M_{L_2}^W = \sum_{X\in\{J,K,L,M,N,P,Q,R,S\}} \left\| \frac{\delta \tilde{H}}{\delta X} \times W_X \right\|_F^2 \quad (15)$$

or

$$M_{L_2}^W = \left\| \frac{\delta \tilde{H}}{\delta Z} \times W_Z \right\|_F^2 \quad (16)$$

The three following propositions are necessary to compute the sensitivity function $M_{L_2}^W$:

Lemma 1. Let consider G and H two matrices (or transfer function) in $\mathbb{C}^{m\times p}$ and $\mathbb{C}^{q\times n}$ and $X \in \mathbb{R}^{p\times q}$. G and H are supposed to be independent with respect to X. Then

$$\frac{\partial (GXH)}{\partial X} = (I_p \otimes G)\frac{\partial X}{\partial X}(I_q \otimes H) \quad (17)$$
$$= G \circledast H \quad (18)$$

and

$$\frac{\partial (GX^{-1}H)}{\partial X} = -(GX^{-1}) \circledast (X^{-1}H) \quad (19)$$

where the $\circledast$ is defined by

$$G \circledast H \triangleq \text{Vec}(G).[\text{Vec}(H^\top)]^\top \quad (20)$$

and Vec *is the usual operator that transforms matrices into column vectors, $\otimes$ is the Kronecker product.*

Proof:
The demonstration is omitted for lack of place. See (Sohl, 2004) or (Neudecker, 1969) for more elements on matrix derivation. ■

Proposition 1. The sensitivity transfer function of $H(z)$ with respect to each matrix of the implicit state-space realization are given by

$$\frac{\partial H}{\partial Z} = \begin{pmatrix} H_3 & H_1 & I_p \end{pmatrix} \circledast \begin{pmatrix} H_4 \\ H_2 \\ I_m \end{pmatrix} \quad (21)$$

$$\frac{\partial D}{\partial Z} = \begin{pmatrix} LJ^{-1} & 0 & I_p \end{pmatrix} \circledast \begin{pmatrix} J^{-1}N \\ 0 \\ I_m \end{pmatrix} \quad (22)$$

with

$$H_1(z) = C(zI_n - A)^{-1} \quad (23)$$
$$H_2(z) = (zI_n - A)^{-1}B \quad (24)$$
$$H_3(z) = H_1(z)KJ^{-1} + LJ^{-1} \quad (25)$$
$$H_4(z) = J^{-1}MH_2(z) + J^{-1}N \quad (26)$$

H_1 and H_2 (that are identical to sensitivity's functions G and F found in (Gevers and Li, 1993)) come from the contribution of X_k in H,

whereas H_3 and H_4 comes from the contribution of T_k in H.

Proof:
The demonstration is omitted for lack of place, but comes from lemma 1, apply on (7). ■

Remark: the SISO case, due to the definition of $\circledast$, leads to same results as in (Hilaire *et al.*, 2005*a*), but expressed in a compact form, thanks to Z.

Then, $M_{L_2^W}$ is computed by applying proposition 2 to equations (21) and (22).

Proposition 2. Let's consider a matrix X, and three transfer functions H, A and B such that

$$\frac{\partial H}{\partial X} = A \circledast B \tag{27}$$

Then, we have

$$\left(\frac{\delta H}{\delta X}\right)_{i,j} = \|A_{\bullet,i} B_{j,\bullet}\|_2 \tag{28}$$

4. DESIGN PROBLEM

Since the parametric sensitivity measure could be evaluated for various equivalent realizations, it could be interesting to find realizations with a maximum tolerance to FWL quantization, *i.e.* realizations with lowest parametric sensitivity measure.
Let $\mathcal{R}_H$ be the set of realizations $\mathcal{R}$ with H as transfer function. The optimal design problem consists in finding the best realization $\mathcal{R}^{opt}$ for the transfer function H according to a measure M (it can be the parametric sensitivity measure or any other measure, like the pole-sensitivity stability related measure)

$$\mathcal{R}^{opt} = \underset{\mathcal{R} \in \mathcal{R}_H}{arg\ min}\ M(R) \tag{29}$$

But this problem is a very difficult one, due to the size of $\mathcal{R}_H$.
A sub-optimal solution of this problem can be found by restricting $\mathcal{R}_H$ to a specific subset, like those defined from a special structuration via a similarity transformation (see (10)).

Then, the parametric sensitivity measure of a realization Z can be computed from the parametric sensitivity measure of the initial realization Z_0 thanks to the following proposition

Proposition 3.

$$\left.\frac{\partial H}{\partial Z}\right|_{Z} = \left(\mathcal{T}_1^{-\top} \otimes I_p\right) \left.\frac{\partial H}{\partial Z}\right|_{Z_0} \left(\mathcal{T}_2^{-\top} \otimes I_m\right)$$

5. NUMERICAL EXAMPLE

The example considered shows how the parametric sensitivity may vary from realizations to another : state-space and observer-based realizations are studied here.
This example is an active control of longitudinal oscillations studied in (Lefebvre *et al.*, 2001) : one significant aspect of vehicle driveability is the attenuation of the first torsional mode (resonance in the elastic parts) which produces unpleasant (0 to 10 Hz) longitudinal oscillations of the car, known as shuffle. They can be reduced by means of a controller acting on the engine torque.
The model of the powertrain was modeled in continuous-time form, and a continuous-time H_∞ optimal controller was designed (Lefebvre *et al.*, 2003). The discretized model $P(z)$ is given by equations (32) and (33), and a discrete-time realization of the controller is given by (33) and (34) : it corresponds to an internally balanced realization.
Remark : all the matrices or results are computed with double floating-point precision, but only 3 significant digital are shown.

Since it exists various ways to implement such a controller, this paper focuses only on classical state-space realizations (shift-operator) and realizations with Observer-State-Feedback forms (realizations with δ-operator, that have proved their numerical efficiency, have been studied in (Gevers and Li, 1993; Hilaire *et al.*, 2005*a*)).

Classical state-space realizations are formalized in the implicit form by considering no temporary variables T_k. The sensitivities of the initial realization (equations (34) and (33)) and the companion one are summarized in the following table. The Adaptive Simulated Annealing (see (Ingber, 1996)) algorithm, a global optimization one, know for its efficiency in the context of control, has been adopted here to search for the optimal realization, according to the sensitivity measure. For this structuration, the similarity transformation used is

$$\mathcal{T}_1 = \begin{pmatrix} I_q & & \\ & T^{-1} & \\ & & I_p \end{pmatrix}, \mathcal{T}_2 = \begin{pmatrix} I_q & & \\ & T & \\ & & I_m \end{pmatrix} \tag{30}$$

This results are coherent with existing ones :

realization	$\mathbf{M_{L_2}^W}$
companion form	$1,78e{+}14$
balanced form	81.44
optimal form	5.99

the canonical form minimizes the execution time, but is very sensitive to the quantization of its parameters. The internally balanced form is quite well numerically conditioned. The optimization process carries on 100 parameters (the coefficients of the matrix T in $\mathcal{T}_1$ and $\mathcal{T}_2$) and took about 4 hours on a desktop computer.

This example was also implemented with a state-feedback-observer structure, particularly because it allows an enrichment of the observer model with a physical meaning but also because these states estimate the states of the physical system. Then, it improves the readability of the signals, and the states initialization of the controller is based on the physical states of the system, so the starting and the commutations to one controller to another (when the gear ratio changes for example) is facilitated. The Observer-State-Feedback is illustrated by equation (31)

$$\left\{ \begin{array}{rcl} \hat{X}_{k+1} & = & A_p\hat{X}_k + B_pU_k \\ & & +K_f(Y_k - C_p\hat{X}_k) \\ U_k & = & -K_c\hat{X}_k + Q(Y_k - C\hat{X}_k) \end{array} \right. \quad (31)$$

The transformation from the state-space form to the Observer-State-Feedback form required to solve a generalized Ricatti equation (Alazard and Apkarian, 1999). The controller poles must be classified between three categories, which are the observation gain, the filter gain and the Youla parameter (static here) : the unobservable pole must be assigned to the estimation gain, the uncontrollable poles must be assigned to the state-feedback gain, and the complex conjugate poles must not be separated (to preserve the gain real). Then, for the other poles, the fast ones are usually (but not necessary) assigned to the estimation gain, and those closed to the physical system to the state-feedback gain. That repartition determines the parameters K_f, K_c and Q.

According to that rules, it is possible to numerically implement equation (31) in various ways, depending on

- the choice of the partition of the poles
- the form of the computation : one way could be to keep $(A_p - K_fC_p)$, B_p, $(K_c + QC_p)$, Q and K_f as parameters (*i.e.* coefficients really implemented in the algorithm). But, it is also possible to choose A_p, B_p, C_p, K_c, K_f and Q as parameters.

Equations (35) and (36) represents the two last possibilities embedded in the Implicit State-Space formalism.

The optimal design consists here in a discrete optimization : all the possible partitions are examined. With 20 poles, it exists 184756 partitions, but only 140 ones are in accordance with the previous rules. The following figure exhibits the sensitivity measure ($log_{10}\left(M_{L_2}^W\right)$ more precisely) of each one, on the first observer-state-feedback form (equation (35)) : the measure varies from $1.358e+2$ to $3.797e+8$. The second observer-state-feedback form (equation (36)) leads to results with same order of magnitude (from $1.423e+2$ to $3.798e+8$).

This example shows that it exists a large diversity of numerical conditioning in the different potential observer-state-feedback realizations, and they

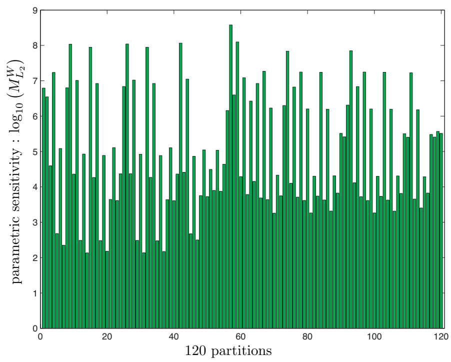

Fig. 1. Parametric sensitivity of each partition examined

have to be taken in consideration in the choice of the poles repartition. Moreover, among the usual partitions, some of them present low sensitivity, *but* some present very bad sensitivity. This criterion must be taken into consideration during the implementation process.

The two observer-state-feedback forms presented have coherent sensitivity : the best partitions for the first form are also the best for the second form. The first form however uses less parameters, whereas in the second one, the parameters A_p, B_p and C_p could be considered as exactly implemented when the quantization is lower than the uncertainties on the plant coefficients. Only parameters K_c, K_f and Q are considered to be approximately implemented : it induces a lower sensitivity.

6. CONCLUSION

The implicit state-space framework allows the macroscopic description of various control algorithms to be implemented. It encapsulates all classical state-space realizations using shift or δ-operators, but also observer-state-feedback ones. This paper has shown how optimisation tool, such as Adaptative Simulated Annealing may be used to find optimal realizations according to the parametric sensitivity measure, well suited for FWL implementation. Moreover, the observer-state-space realizations have their own interests and it has been shown that the degree of freedom associated to such a realization (observer dynamics versus state feedback one) may be pertinently used to obtain, by discrete optimization, a low sensitivity measure.

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REFERENCES

Alazard, D. and P. Apkarian (1999). Exact observer-based structures for arbitrary compensators. *International Journal of Robust and Non-Linear Control* (9), 101–118.

Chen, S., J. Wu and G. Li (2002). Two approaches based on pole sensitivity and stability radius measures for finite precision digital controller realizations. *Systems and Control Letters* **45**(4), 321–329.

Gevers, M. and G. Li (1993). *Parametrizations in Control, Estimation and Filtering Probems*. Springer-Verlag.

Hilaire, T., P. Chevrel and Y. Trinquet (2005*a*). Designing low parametric sensitivity FWL realizations of LTI controllers/filters within the implicit state-space framework. In: *CDC-ECC'05*.

Hilaire, T., P. Chevrel and Y. Trinquet (2005*b*). Implicit state-space representation : a unifying framework for FWL implementation of LTI systems. In: *IFAC05 World Congress*.

Ingber, L. (1996). Adaptive simulated annealing (asa): Lessons learned. *Control and Cybernetics* **25**(1), 33–54.

Lefebvre, D., P. Chevrel and S. Richard (2001). An h-infinity based control design methodology dedicated to active control of longitudinal oscillations. In: *Proceedings of the Conference on Decision and Control*.

Lefebvre, D., P. Chevrel and S. Richard (2003). An h-infinity based control design methodology dedicated to the active control of longitudinal oscillations. *IEEE Transactions on Control Systems Technology* **11**(6), 948–956.

Li, G. (1998). On the structure of digital controllers with finite word length consideration. In: *IEEE Transactions on Automatic Control*. Vol. 43. pp. 689–693.

Neudecker, H. (1969). Some theorems on matrix differentiation with special reference to kronecker matrix products. *Journal of American Statistical Association* **64**, 953–963.

Sohl, C. (2004). Kronecker tensor products, differentiation theory for matrices and applications. Master's thesis. Lund Institute of Technology.

$$A_p=\begin{pmatrix}
8.384e{-1} & 1.600e{-1} & -3.294e{-1} & -4.833e{-2} & 0 & 0 & 0 & 0 & 0 & 0\\
-3.927e{-1} & 7.144e{-1} & 5.040e{-2} & -8.245e{-3} & 0 & 0 & 0 & 0 & 0 & 0\\
-1.566e{-1} & -6.105e{-1} & 3.683e{-2} & 4.195e{-1} & 0 & 0 & 0 & 0 & 0 & 0\\
-1.444e{-1} & 1.772e{-1} & -6.798e{-1} & 6.508e{-1} & 0 & 0 & 0 & 0 & 0 & 0\\
1.929e{-1} & 1.512e{-1} & 4.030e{-1} & 3.898e{-1} & 9.773e{-1} & 1.037e{-2} & -6.170e{-2} & 0 & 0 & 0\\
2.768e{-4} & 2.170e{-4} & 5.783e{-4} & 5.594e{-4} & 2.837e{-3} & 9.971e{-1} & 1.698e{-2} & 0 & 0 & 0\\
3.238e{-2} & 2.539e{-2} & 6.767e{-2} & 6.545e{-2} & 3.320e{-1} & -3.341e{-1} & 9.868e{-1} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000e{+0} & -1.000e{-10} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000e{-2} & 1.000e{+0} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9.417e{-1}
\end{pmatrix} \tag{32}$$

$$B_p=\begin{pmatrix} -4.007e{+0}\\ -5.769e{+0}\\ -6.522e{+0}\\ 2.490e{+0}\\ 8.562e{-1}\\ 1.229e{-3}\\ 1.438e{-1}\\ 1.000e{+0}\\ 5.000e{-3}\\ 0 \end{pmatrix},\ C_p=\begin{pmatrix} 9.209e{-3}\\ 7.221e{-3}\\ 1.924e{-2}\\ 1.861e{-2}\\ 9.441e{-2}\\ 4.953e{-4}\\ -2.946e{-3}\\ 0\\ 0\\ -3.495e{-1} \end{pmatrix}^{\top},\ B=\begin{pmatrix} -2.372e{+0}\\ -2.540e{+0}\\ -1.210e{-1}\\ -1.565e{-4}\\ -6.245e{-2}\\ 1.151e{+0}\\ 4.083e{-2}\\ 2.255e{-1}\\ -1.528e{-2}\\ -9.720e{-4} \end{pmatrix},\ C=\begin{pmatrix} -2.372e{-2}\\ 2.540e{-2}\\ 1.210e{-3}\\ -1.565e{-6}\\ 6.245e{-4}\\ 1.151e{-2}\\ 4.083e{-4}\\ -2.255e{-3}\\ 1.528e{-4}\\ 9.720e{-6} \end{pmatrix}^{\top},\ D=-2.140e{-1} \tag{33}$$

$$A=\begin{pmatrix}
8.195e{-1} & 2.812e{-1} & -3.317e{-2} & 2.699e{-2} & -1.649e{-1} & 1.318e{-1} & 1.059e{-2} & -6.733e{-2} & 1.750e{-3} & 6.525e{-5}\\
-2.812e{-1} & -4.817e{-1} & -1.668e{-1} & 8.654e{-2} & -5.403e{-1} & 1.469e{-1} & 1.837e{-2} & -1.211e{-1} & 1.942e{-3} & 2.134e{-5}\\
3.317e{-2} & -1.668e{-1} & 9.749e{-1} & 1.696e{-2} & -9.104e{-2} & 7.638e{-2} & 3.357e{-3} & -2.006e{-2} & 8.441e{-4} & 4.548e{-5}\\
2.699e{-2} & -8.654e{-2} & -1.696e{-2} & 9.601e{-1} & 2.528e{-1} & 5.956e{-2} & 1.654e{-3} & -9.085e{-3} & 6.046e{-4} & 3.843e{-5}\\
1.649e{-1} & -5.403e{-1} & -9.104e{-2} & -2.528e{-1} & 6.022e{-1} & 3.888e{-1} & 1.150e{-2} & -6.420e{-2} & 3.945e{-3} & 2.454e{-4}\\
1.318e{-1} & -1.469e{-1} & -7.638e{-2} & 5.956e{-2} & -3.888e{-1} & 4.664e{-1} & -6.206e{-2} & 4.224e{-1} & -8.490e{-4} & 3.703e{-4}\\
1.059e{-2} & -1.837e{-2} & -3.357e{-3} & 1.654e{-3} & -1.150e{-2} & -6.206e{-2} & 9.832e{-1} & 1.258e{-1} & 7.737e{-3} & 6.392e{-4}\\
6.733e{-2} & -1.211e{-1} & -2.006e{-2} & 9.085e{-3} & -6.420e{-2} & -4.224e{-1} & -1.258e{-1} & -4.483e{-2} & 7.258e{-2} & 5.631e{-3}\\
-1.750e{-3} & 1.942e{-3} & 8.441e{-4} & -6.046e{-4} & 3.945e{-3} & 8.490e{-4} & -7.737e{-3} & 7.258e{-2} & 9.838e{-1} & -2.474e{-3}\\
-6.525e{-5} & 2.134e{-5} & 4.548e{-5} & -3.843e{-5} & 2.454e{-4} & -3.703e{-4} & -6.392e{-4} & 5.631e{-3} & -2.474e{-3} & 9.418e{-1}
\end{pmatrix} \tag{34}$$

$$\begin{pmatrix} I & 0 & 0\\ -B_p & I & 0\\ -I & 0 & I \end{pmatrix}\begin{pmatrix} T_{k+1}\\ \hat{X}_{k+1}\\ U_k \end{pmatrix} = \begin{pmatrix} 0 & -(QC_p+K_c) & Q\\ 0 & (A_p-K_fC) & K_f\\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} T_k\\ \hat{X}_k\\ Y_k \end{pmatrix} \tag{35}$$

$$\begin{pmatrix} \begin{pmatrix} I & 0\\ -Q & I \end{pmatrix} & \begin{pmatrix} 0\\ 0 \end{pmatrix} & \begin{pmatrix} 0\\ 0 \end{pmatrix}\\ \begin{pmatrix} -K_f & -B_p \end{pmatrix} & I & 0\\ \begin{pmatrix} 0 & -I \end{pmatrix} & 0 & I \end{pmatrix}\begin{pmatrix} \begin{pmatrix} T^{(1)}_{k+1}\\ T^{(2)}_{k+1} \end{pmatrix}\\ \hat{X}_{k+1}\\ U_k \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0\\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -C_p\\ -K_c \end{pmatrix} & \begin{pmatrix} I\\ 0 \end{pmatrix}\\ \begin{pmatrix} 0 & 0 \end{pmatrix} & A_p & 0\\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & 0 \end{pmatrix}\begin{pmatrix} \begin{pmatrix} T^{(1)}_{k}\\ T^{(2)}_{k} \end{pmatrix}\\ \hat{X}_k\\ Y_k \end{pmatrix} \tag{36}$$