Interval-based Robustness of Linear Parametrized Filters

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Introduction. This article deals with the resilient implementation of parametrized linear filters (or controllers), *i.e.* with realizations that are robust with respect to their fixed-point implementation.
The implementation of a linear filter/controller in an embedded device is a difficult task due to numerical deteriorations in performances and characteristics. These degradations come from the quantization of the embedded coefficients and the roundoff occurring during the computations.

As mentioned in [1], there are an infinity of equivalent possible algorithms to implement a given transfer function h. To cite a few of them, one can use direct forms, state-space realizations, ρ-realizations, etc. Although they do not require the same amount of computation, all these realizations are equivalent in infinite precision, but they are no more in finite precision. The *optimal realization problem* is then to find, for a given filter, the most resilient realization.

We here consider an extended problem with filters those coefficients depend on a set $\boldsymbol{\theta}$ of parameters that are not exactly known during the design. They are used for example in automotive control, where a very late fine tuning is required.

Linear parametrized filters. Following [3], we denote $\boldsymbol{Z}(\boldsymbol{\theta})$ the matrix containing all the coefficients used by the realization, $h_{\boldsymbol{Z}(\boldsymbol{\theta})}$ the associated transfer function and $\boldsymbol{\theta}^\dagger$ the quantized version of $\boldsymbol{\theta}$. $\boldsymbol{Z}^\dagger(\boldsymbol{\theta}^\dagger)$ is then the set of the quantized coefficients, *i.e.* the quantization of coefficients $\boldsymbol{Z}(\boldsymbol{\theta}^\dagger)$ computed from the quantized parameters $\boldsymbol{\theta}^\dagger$. The corresponding transfer function is denoted $h_{\boldsymbol{Z}^\dagger(\boldsymbol{\theta}^\dagger)}$.

Performance Degradation Analysis. The two main objectives of this article are to evaluate the impact of the quantization of $\boldsymbol{\theta}$ and $\boldsymbol{Z}(\boldsymbol{\theta})$ on the filter performance and to estimate the parameters $\boldsymbol{\theta}$ that give the worst transfer function error in the set of possible parameters $\boldsymbol{\Theta}$.

For that purpose, there are mainly two kinds of tools to study the degradation of filter performance due to the quantization effect: *i)* use a sensitivity measure (with respect to the coefficients) based on a first order approximation and a statistical quantification error model; *ii)* use interval tool, based on transfer function with interval coefficients. In both cases, we seek the maximal distance between the exact transfer function $h_{\boldsymbol{Z}(\boldsymbol{\theta})}$ and the quantized one $h_{\boldsymbol{Z}^\dagger(\boldsymbol{\theta}^\dagger)}$. For that purpose, we can use the *L_2-norm i.e.*, $\| g \|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_0^{2\pi} | g(e^{j\omega}) |^2 \, d\omega}$ or the *Maximum norm i.e.*, $\| g \|_\infty \triangleq \max_{\omega\in[0,2\pi]} | g(e^{j\omega}) |$.

The measure of the degradation of the finite precision implementation is then given by $\| h_{\boldsymbol{Z}(\boldsymbol{\theta})} - h_{\boldsymbol{Z}^\dagger(\boldsymbol{\theta}^\dagger)} \|_\diamond$, with $\diamond \in \{2, \infty\}$. So the worst-case parameters $\boldsymbol{\theta}_0$ can be found by solving:

$$\arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \| h_{\boldsymbol{Z}(\boldsymbol{\theta})} - h_{\boldsymbol{Z}^\dagger(\boldsymbol{\theta}^\dagger)} \|_\diamond \quad . \tag{1}$$

Since $\boldsymbol{\Theta}$ is an interval vector, we denote $[h]$ the interval transfer function. With an interval approach, we can define the following constrained global optimization problem:

$$\text{Maximize } \| [h]^\dagger_{\boldsymbol{Z}^\dagger(\boldsymbol{\theta}^\dagger)} - [h]_{\boldsymbol{Z}(\boldsymbol{\theta})} \|_\diamond \quad \text{subject to} \quad \boldsymbol{\theta} \in \boldsymbol{\Theta} \quad . \tag{2}$$

Note that in both cases, the evaluation of the norms can be done in interval with $\omega \in [0, 2\pi]$.

We will present the solutions of this problem using interval optimization methods [2] and we will compare them with the statistical sensitivity approach.

References:

[1] H. Hanselmann. Implementation of digital controllers - a survey. *Automatica*, 23(1):7–32, January 1987.

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[3] T. Hilaire, P. Chevrel, and J.F. Whidborne. A unifying framework for finite wordlength realizations. *IEEE Trans. on Circuits and Systems*, 8(54):1765–1774, August 2007.