Evaluation of the Root Mean Square Error Performance of the PAST-Consensus Algorithm

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***Abstract*—In previous work, we developed and investigated a distributed Projection Approximation Subspace Tracking Algorithm (PAST-Consensus) based on Consensus Propagation for wireless sensor networks. Preliminary simulation results showing a good tracking capability and still reduced complexity, have motivated us to evaluate the performance of the aforementioned algorithm. In this work, some simulation results will be presented comparing the root mean square error for several signal to noise ratios, as well as the error in the signal subspace given by its angle difference.**

I. Introduction

Subspace methods have become a very important subject of study in contemporary signal processing. There are several subspace estimation algorithms such as MUSIC, ESPRIT, and ESF which calculate for instance, the direction of arrivals (DOA) of plain waves impinging on a sensor array or estimate the frequencies of sinusoids that lie in a specific space [1], [2], [3], [4], [5].

However, these traditional methods present a high complexity due to the fact that they rely on eigenvalue/singular value decomposition for estimating the signal or the noise subspace. These algorithms have to calculate the sample covariance matrix every time a new sample arrives, which leads to a high amount of internal processing.

The goal of this paper is to evaluate the performance of the PAST-Consensus algorithm developed in [6]. This method is a distributed version of the Projection Approximation Subspace Tracking (PAST) [7], a well-known algorithm whose major advantage is the considerably low complexity. PAST introduces a new signal subspace model interpretation and presents a solution where the signal subspace is calculated in a recursive manner at time t, depending on the previous subspace estimate at time $t-1$ and the new sample $\underline{\boldsymbol{x}}(t)$.

Likewise, Consensus Propagation is a remarkable protocol widely used for obtaining averages over a network [8], [9], [10]. The work in [6] has introduced a distributed algorithm based on Consensus Propagation for frequency estimation over a wireless sensor network.

Notation: We denominate a column vectors in underlined boldface and matrices in uppercase and boldface. The superscript ^{H} represents complex conjugation, $^\top$ transposition, $tr(.)$ the trace operator and $\|.\|$ the Euclidean vector norm.

Organization of the paper: Section II presents an overview of the analytical model and constraints. The section III introduces the PAST-Consensus algorithm as implemented in [6]. Finally, in Section IV we will present some simulation results that evaluate the root mean square error for several signal to noise ratios, as well as the signal subspace error expressed in terms of principal angle components.

II. An overview of the Projection Approximation Subspace Tracking algorithm

Let $\underline{\boldsymbol{x}}(t) \in \mathbb{C}^N$ be the data vector observed at time t, composed of r narrow-band signal waves impinging on a planar array of N sensors hidden in additive noise (zero-mean with variance σ^2). The data vector is modelled as

$$\begin{aligned}\underline{\boldsymbol{x}}(t) &= \sum_{i=1}^{r} s_i(t)\underline{\boldsymbol{a}}(w_i) + \underline{\boldsymbol{n}}(t) \\ &= \boldsymbol{A}(\underline{\boldsymbol{\omega}})\underline{\boldsymbol{s}}(t) + \underline{\boldsymbol{n}}(t)\end{aligned} \tag{1}$$

with,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ \mathrm{e}^{j\omega_1} & \mathrm{e}^{j\omega_2} & \mathrm{e}^{j\omega_r} \\ \vdots & \vdots & \vdots \\ \mathrm{e}^{(N-1)j\omega_1} & \mathrm{e}^{(N-1)j\omega_2} & \mathrm{e}^{(N-1)j\omega_r} \end{pmatrix},$$

$$\underline{\boldsymbol{s}}(t) = \begin{pmatrix} s_1(t) \\ \vdots \\ s_r(t) \end{pmatrix} \text{ and } \underline{\boldsymbol{n}}(t) = \begin{pmatrix} n_1(t) \\ \vdots \\ n_r(t) \end{pmatrix}.$$

Here, $\boldsymbol{A}(\underline{\boldsymbol{\omega}})$ is a deterministic $N \times r$ signal mixing matrix depending on $\underline{\boldsymbol{\omega}} = (\omega_1, \ldots, \omega_r)$. Its columns are plane wave steering vectors $\underline{\boldsymbol{a}}(\omega_i)$ and the n-th element of $\underline{\boldsymbol{a}}(\omega)$ is defined by

$$[\underline{\boldsymbol{a}}(\omega_i)]_n = \exp\left(\mathrm{j}\frac{2\pi}{\lambda}(\xi_n \cos\theta_i + \eta_n \sin\theta_i)\right)/\sqrt{N}$$

where $\omega_i = \cos\theta_i$ are the frequency that lies in the subspace of $\boldsymbol{A}$ and that we intend to track. The Cartesian coordinates of the n-th sensor node are (ξ_n, η_n) and λ is the wavelength,

$\underline{\boldsymbol{s}}(t)$ is the vector of complex signal amplitudes and $\underline{\boldsymbol{n}}(t)$ is uncorrelated additive noise. We allow $\underline{\boldsymbol{\omega}}$ to be slowly time-varying and seek an estimate for matrix $\boldsymbol{W}(t)$ whose columns span the same subspace as $\boldsymbol{A}(\underline{\boldsymbol{\omega}}(t))$. A tracking algorithm estimates $\boldsymbol{W}(t)$ by a function of the previously estimated matrix $\boldsymbol{W}(t-1)$ and the new observation $\underline{\boldsymbol{x}}(t)$ alone. Particularly, Yang [7] approximates the cost function to minimize

$$J(\boldsymbol{W}(t)) = \sum_{i=1}^{t} \beta^{t-i} \left\| \underline{\boldsymbol{x}}(i) - \boldsymbol{W}(t)\boldsymbol{W}^H(t)\underline{\boldsymbol{x}}(i) \right\|^2 \quad (2)$$

by

$$J'(\boldsymbol{W}(t)) = \sum_{i=1}^{t} \beta^{t-i} \left\| \underline{\boldsymbol{x}}(i) - \boldsymbol{W}(t)\underline{\boldsymbol{y}}(i) \right\|^2 \quad (3)$$

with

$$\underline{\boldsymbol{y}}(i) = \boldsymbol{W}^H(i-1)\underline{\boldsymbol{x}}(i) \; . \quad (4)$$

Here, the vector $\underline{\boldsymbol{y}}(t)$ stores the modified data vector resulting from the new incoming data $\underline{\boldsymbol{x}}(t)$ and the previous estimated signal subspace $\boldsymbol{W}(t)$. Besides, the matrix $\boldsymbol{W} \in \mathbb{C}^{N \times r}$ is constrained to rank $r < N$ and the minimization of cost function (3) results in a low-complexity update of the signal subspace.

III. PAST-Consensus

A. *Consensus Propagation*

In the following, consider a wireless sensor network composed of $N = 36$ nodes placed in an irregular cartesian grid, as shown for instance in Figure 1. The average distance between nodes is $\lambda/2$ and the transmission range is set to $1.44\lambda/2$. For every node $n = 1, \ldots, N$ in the network, $\mathcal{N}_n$ denominates the set of the adjacent nodes including itself. In this setting each node in the neighborhood $\mathcal{N}_n$ can correctly receive broadcasted messages from nodes within its neighborhood with probability 1. One can observe on **Algorithm 1** that at the end of step $t-1$, every node n sends its own estimation of the average $\underline{\boldsymbol{y}}_n(t-1)$ and a weight w_n to its adjacent nodes $\mathcal{N}_n$. Then, at the beginning of step t, every node n receives the pairs $\{\underline{\boldsymbol{y}}_i(t-1), w_i\}_{i\in\mathcal{N}_n}$ from its neighbors and compute a new average, to be sent at the end of step t.

The aforementioned weights are constant for every node and defined as

$$w_n = 1/\sqrt{|\mathcal{N}_n|} \;\; \forall 1 \le n \le N. \quad (5)$$

Finally, the local observation vector $\underline{\boldsymbol{x}}_n(t)$ is obtained with the aggregation of data observed by the adjacent nodes, denoted by $\{x_j(t-1)\}_{j\in\mathcal{N}_n}$.

B. *PAST-Consensus propagation algorithm*

The distributed subspace tacking algorithm presented in [6], has been conceived for sensor network applications. Due to its low computational complexity $O(nr)$, scalability and robustness, it seems a suitable approach for distributed sensor network applications.

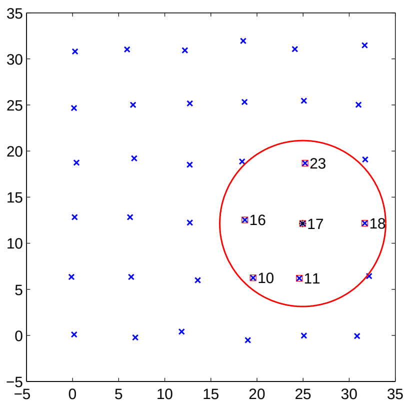

Fig. 1. Sensor network with neighborhood $\mathcal{N}_{17}$ for radius 9.

Algorithm 1: Consensus Propagation Algorithm

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for $n := 1, 2, \ldots, N$ **do**

  Input: $\{\underline{\boldsymbol{y}}_j(t-1), w_j\}_{j\in\mathcal{N}_n}$ are the pairs sent to node n in step $t-1$

$$\underline{\boldsymbol{y}}_n(t) = \left(\sum_{j\in\mathcal{N}_n} \underline{\boldsymbol{y}}_j(t-1) w_j \right) / \left(\sum_{j\in\mathcal{N}_n} w_j \right) \quad (6)$$

  Broadcast the pair $\{\underline{\boldsymbol{y}}_n(t), w_n\}$ to all nodes in $\mathcal{N}_n$

  Output: $\underline{\boldsymbol{y}}_n(t)$ is the estimation of the average in step t at node n

endfor

Sensor systems are well known for a wide range of applications such as the monitoring of environmental phenomena, i.e. avalanches, floods. Constant improvement is sought in the areas of power consumption, internal processing load, scalability and robustness.

The PAST-Consensus algorithm is an alternative for tackling some of this problems. In addition, it locally averages the vector $\underline{\boldsymbol{y}}_n(t)$ in n with information from $\mathcal{N}_n$. Afterwards, n broadcasts its local observation $x_n(t)$, the locally filtered r-dimensional vector $\underline{\boldsymbol{y}}_n(t)$, and a weight w_n. To conclude, $\underline{\boldsymbol{y}}_n(t)$ in equation (4) contains information from the updated signal subspace at $t-1$ as well as new observation data $\underline{\boldsymbol{x}}_n(t)$. The aforementioned algorithm is given by **Algorithm 2**.

In the following, we will show some simulation results which analyze the performance of the algorithm in terms of the root mean square error and calculate the error between the original and the estimated subspace at node n. Here, the distance between two subspaces is equal to two times the sum of the squared sines of the principal angles [11].

IV. Simulations

In order to make a consistent data performance evaluation, we propose the following simulation scenario. Consider a planar sensor network with $N = 36$ nodes, whose position resembles an imperfect Cartesian grid. The exact sensor array geometry is documented in [6] and represented in **Figure 1**.

Algorithm 2: PAST-Consensus [6]

Input: β, $\boldsymbol{P}_1(0),\ldots,\boldsymbol{P}_N(0)$, $\boldsymbol{W}_1(0),\ldots,\boldsymbol{W}_N(0)$
for $t := 1, 2, \ldots$ **do**
  for $n := 1, 2, \ldots, N$ **do**
    Input: $x(n)$
    aggregate $\underline{\boldsymbol{x}}_n(t) = \boldsymbol{S}_n \underline{\boldsymbol{x}}(t-1)$ from all nodes$\in \mathcal{N}_n$
    $\underline{\boldsymbol{y}}_n(t) = \boldsymbol{W}_n^H(t-1)\underline{\boldsymbol{x}}_n(t)$
    apply **Algorithm 1** for locally averaging $\underline{\boldsymbol{y}}_n(t)$
    $\underline{\boldsymbol{h}}_n(t) = \boldsymbol{P}_n(t-1)\underline{\boldsymbol{y}}_n(t)$
    $\underline{\boldsymbol{g}}_n(t) = \underline{\boldsymbol{h}}_n(t)/[\beta + \underline{\boldsymbol{y}}_n^H(t)\underline{\boldsymbol{h}}_n(t)]$
    $\boldsymbol{P}_n(t) = \frac{1}{\beta}[\boldsymbol{P}_n(t-1) - \underline{\boldsymbol{g}}_n(t)\underline{\boldsymbol{h}}_n^H(t)]$
    $\underline{\boldsymbol{e}}_n(t) = \boldsymbol{D}_n(\underline{\boldsymbol{x}}(t) - \boldsymbol{W}_n(t-1)\underline{\boldsymbol{y}}_n(t))$
    $\boldsymbol{W}_n(t) = \boldsymbol{W}_n(t-1) + \underline{\boldsymbol{e}}_n(t)\underline{\boldsymbol{g}}_n^H(t)$
    broadcast $\{x_n(t), \underline{\boldsymbol{y}}_n(t), w_n\}$ to all nodes$\in \mathcal{N}_n$
  endfor
endfor

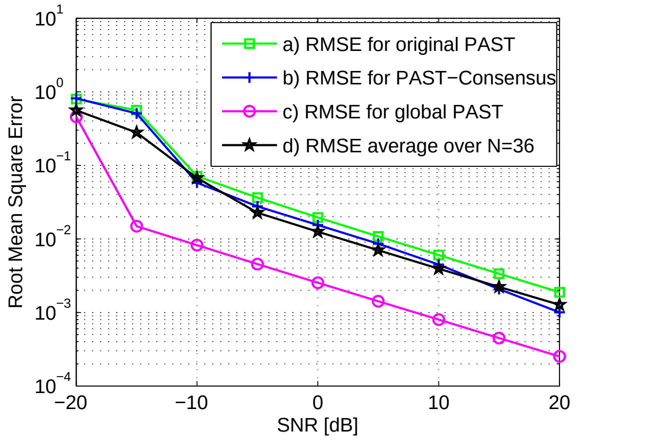

Fig. 2. Root Mean Square Error comparison for $r = 1$: a) PAST result for $\mathcal{N}_{17}, N = 6$, b) PAST-Consensus result for neighborhood $\mathcal{N}_{17}, N = 6$, c) PAST result for whole sensor array $N = 36$

Here, every sensor network observes 1000 snapshots in time $(t = 1, ..., 1000)$ and tracks the instantaneous frequency $\omega_1(t) = 0.1$ of a single impinging wave which is constant over time. The forgetting factor is chosen to be $\beta = 0.97$, $\boldsymbol{W}(0)$ and $\boldsymbol{P}(0)$ are initialized as identity matrices. We assume that messages sent by node n are received correctly by all nodes in its neighborhood $\mathcal{N}_n$. The spectral MUSIC algorithm is used for extracting the $r = 1$ frequency component $\hat{\omega}_1(t)$ from the signal subspace matrix $\boldsymbol{W}(t)$.

In the first experiment, we consider the performance at node 17 and compare the behavoir of the PAST-Consensus algorithm (see **Figure 2**, curve b)) with:

a) the original PAST-algorithm in [7], where only node 17 and its neighborhood $\mathcal{N}_{17}$ are considered $(N = 6)$.
c) the original PAST algorithm applied to the *global* sensor network. Here, we suppose that each node can simultaneously access all nodes in the network, and we keep using the non-distributed version of [7]. Again, the starting point for such calculations is the node 17.
d) the average RMSE when considering all nodes in the network $N = 36$.

In our simulations, the root mean square error formula at each node for one single frequency $(r = 1)$ is defined as:

$$\text{RMSE\_}n = \sqrt{\frac{1}{901}\sum_{t=100}^{1000} |\omega_{1,n}(t) - \hat{\omega}_{1,n}(t)|^2}, \tag{7}$$

where $\omega_{1,n}(t)$ is the true frequency value and $\hat{\omega}_{1,n}(t)$ its estimate. Since the **Algorithm 2** does not require any a priori data statistics, the initial information tends to be the least reliable. Due to this fact, we neglect the first one hundred snapshots in our graphs in order to obtain a more representative error sampling.

For the different simulations, we consider the additive noise $\underline{\boldsymbol{n}}(t)$ (see eq. (1)) with a Signal Noise Ration (SNR) from -20dB to 20dB. The simulation runs are carried out with the same noise realization.

Note that this definition of RMSE_n is the root of the time averaged square error, which is different from the classical definition based on the ensemble average of the square error.

In **Figure 2**, we observe the performance at node 17 for different signal to noise ratios. In a) the original PAST algorithm is analyzed for the neighborhood $\mathcal{N}_{17}$ $(N = 6$, $r = 1)$. Likewise, in b) we examine the root mean square error (RMSE) for the same neighborhood, but employing the **PAST-Consensus** based on Consensus Propagation. Both errors a) and b) behave the same way at the beginning, since the initialization parameters for the algorithms were the same. However, our distributed approach seems to outperform the centralized solution after -19dB, where it stays constant. Note that for SNR values higher than 10 dB the error factor decays faster. As expected in c), the error evaluation for n=17 considering the whole neighborhood N= 36 has the best performance, since all nodes have access to all data available in the sensor network.

In the second experiment, we consider again 1000 iterations for the case of two crossing frequencies, ω_1=-0.5 to 0.5 and ω_2=0.5 to -0.5. The simulation runs are carried out with the same noise realization. However, the SNR for the image a) is set to 5dB and for b) the SNR= -10dB. We observe keep simulating for the node 17.

Figure 3 shows the principal angles between the subspace spanned by the columns of the estimated signal subspace $\boldsymbol{W}_n(t)$ and the original signal subspace spanned by the columns of the matrix $\boldsymbol{A}$, as in (1). Notice that the graph a) shows a good tracking capability for a signal to noise ratio set to 5dB. However, in the graph b) the estimation accuracy diminishes for a SNR=-10dB. The principal angles are zero if the subspaces compared are the same. However, this is not case for the graphs c) and d), where the angles never reach zero. Nevertheless, one can observe that the evolution

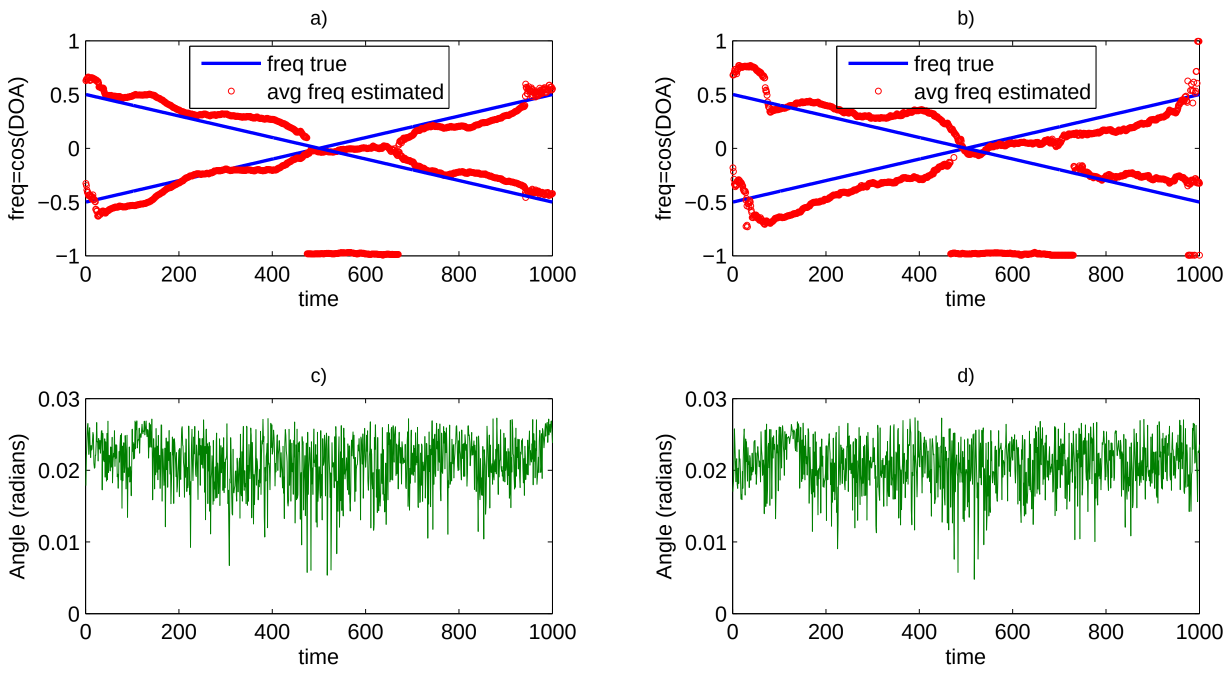

Fig. 3. Figures a) and b): Two crossing frequencies (r=2, n=17) showing the tracking capabilities of the PAST-Consensus algorithm for SNR=5dB and SNR=-5dB respectively. Figures c) and d): Distance between two subspaces expressed in terms of its principal angles. Namely, the signal subspace matrix $\boldsymbol{W_n}(t)$ and the signal mixing matrix $\boldsymbol{A}$ of $\mathcal{N}_{17}$ (see eq. (1)). Again, the calculations are for $n = 17$, $SNR = 5dB$ and $SNR = -5dB$ correspondingly

of the error in the subspace seems not to be dramatically affected for different SNR values. The reason for this is that the PAST-Consensus algorithm, as previously mentioned, estimates the signal subspace in a recursive manner and does not need any knowledge of the eigenvalues. The algorithms based on eigenvalue decomposition are very sensitive to low SNR values, because the gap separating the signal from the noise subspace is reduced and is more difficult to make a correct identification of both.

V. Summary and Conclusion

We have calculated the distance between the estimated signal subspace $\boldsymbol{W}(t)$ and the original subspace introduce by the matrix $\boldsymbol{A}$, as in (1) in terms of the principal angles. The results are quite interesting: On one hand the error between subspaces does not lead to zero, it rather stays constant over time for values between 0.03 and 0.01 radians. On the other hand, one can observe that the signal subspace interpretation in [7] leads to an algorithm more robust against errors, when considering too closely spaced or very weak signals.

The RMSE simulation result indicates that the **PAST-Consensus** algorithm has a performance in terms of the RMSE between the (centralized) PAST algorithm [7] which uses the data of *all* sensors and a PAST algorithm which uses only the sensor data of the immediately neighboring nodes. In future research, we will optimize the weighting coefficients used for consensus propagation which will further reduce the RMSE of the **PAST-Consensus** algorithm while conserving its good tracking capability and low complexity.

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