Pole Sensitivity Stability Related Measure of FWL Realizations with the Implicit State-Space Formalism

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Context

- Implementation of Linear Time Invariant controllers
- Finite Word Length context

Motivation

- Evaluate the impact of the quantization of the embedded coefficients
- Compare various realizations and find an *optimal* one

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Origin of the degradation

The deterioration induced by the FWL implementation comes from :

- Quantization of the involved coefficients
  $\longrightarrow$ *parametric errors*
- Roundoff noises in numerical computations
  $\longrightarrow$ *numerical noises*

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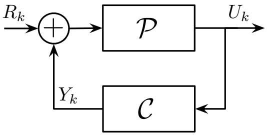
Problem setup

Let's consider a discrete plant $\mathcal{P}$

$$\mathcal{P}\left\{\begin{array}{rcl} X^p_{k+1} &=& A_p X^p_k + B_p(R_k + Y_k) \\ U_k &=& C_p X^p_k \end{array}\right.$$



and a LTI controller $\mathcal{C}$

$$\mathcal{C}\left\{\begin{array}{rcl} X_{k+1} &=& AX_k + BU_k \\ Y_k &=& CX_k + DU_k \end{array}\right.$$

The realizations of the form $(T^{-1}AT, T^{-1}B, CT, D)$, with T a non-singular matrix, are all equivalent in infinite precision.
They are no more in finite precision.
The degradation of the realization depends on the realization.

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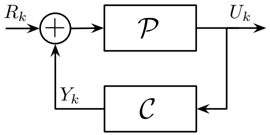

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Problem setup

When quantized, the parameters $X \triangleq \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ are changed in $X + \Delta X$ and the closed-loop system can became unstable.

Let's denote $\left(\lambda_k(\bar{A}(X))\right)_{1\leqslant k\leqslant l}$ the eigenvalues of the closed-loop system

$$\begin{cases} \bar{X}_{k+1} = \bar{A}\bar{X}_k + \bar{B}R_k \\ U_k = \bar{C}\bar{X}_k \end{cases} \qquad (1)$$

with

$$\bar{A} \triangleq \begin{pmatrix} A_p + B_p D C_p & B_p C \\ B C_p & A \end{pmatrix} \qquad (2)$$
$$\bar{B} \triangleq \begin{pmatrix} B_p \\ 0 \end{pmatrix} \bar{C} \triangleq \begin{pmatrix} C_p & 0 \end{pmatrix}$$

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Pole-sensitivity stability related measure

Chen, Wu, Li,... (2000-2005) have proposed a pole-sensitivity measure defined by

$$\mu(X) \triangleq \min_{1\leqslant k\leqslant r} \frac{1-\left|\lambda_k(\bar{A}(X))\right|}{\sqrt{N\Psi_k}}$$

with N is the number of non-trivial elements in X (non-zero elements in ΔX) and Ψ_k is the pole sensitivity of the closed-loop with respect to the parameters :

$$\Psi_k \triangleq \sum_{i,j} (W_X)_{i,j} \left| \frac{\partial \left|\lambda_k(\bar{A}(X))\right|}{\partial X_{i,j}} \right|^2$$

W_X is the weighting matrix associated to the realization matrix X, defined by

$$(W_X)_{i,j} = \begin{cases} 0 & \text{if } X_{i,j} \text{ is \textit{exactly} implemented} \\ 1 & \text{if not} \end{cases}$$

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- the measure is such that

$$\|\Delta X\|_{\max} \leqslant \mu(X) \Rightarrow \bar{A}(X+\Delta X) \text{ is stable}$$

- it considers how close the eigenvalues are to 1 and how sensitive they are w.r.t the controller parameters ;
- this measure is directly linked an estimation of the smallest word-length bit needed to guarantee the closed-loop stability
- the *optimal design problem* associated consists in finding an equivalent realization $(T^{-1}AT, T^{-1}B, CT, D)$ that maximizes this measure.

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The need of a unifying framework

Various implementation forms have to be taken into consideration

- shift-realizations
- δ-realizations
- observer-state-feedback
- direct form I or II
- cascade or parallel realizations
- etc...

The need of a unifying framework

In order to encompass all these implementations, we have proposed a specialized implicit state-space realization to be used as a unifying framework :

Interests

- macroscopic description of a FWL implementation
- more general than previous realizations
- more realistic with regard to the parameterization
- directly linked to the in-line computations to be performed

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Implicit State-Space Framework

The control algorithm is described with

1. $J.T_{k+1} = M.X_k + N.U_k$
2. $X_{k+1} = K.T_{k+1} + P.X_k + Q.U_k$
3. $Y_k = L.T_{k+1} + R.X_k + S.U_k$

Intermediate variables computation

Implicit State-Space Framework

$$\begin{pmatrix} J & 0 & 0 \\ -K & I & 0 \\ -L & 0 & I \end{pmatrix} \begin{pmatrix} T_{k+1} \\ X_{k+1} \\ Y_k \end{pmatrix} = \begin{pmatrix} 0 & M & N \\ 0 & P & Q \\ 0 & R & S \end{pmatrix} \begin{pmatrix} T_k \\ X_k \\ U_k \end{pmatrix}$$

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Output computation

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Intermediate variables

The intermediate variables introduced allow to

- make explicit all the computations done
- show the order of the computations
- express a larger parameterization

All the coefficients used in the implicit framework can be regrouped in a *generalized system matrix Z*

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Implicit form

This form is implicit (but non singular) :

- the state or the output may be computed from intermediate variables
- an intermediate variable may be computed from another intermediate variable previously computed (in the same step)
  computation of T_{k+1} is $J.T_{k+1} = M.X_k + N.U_k$

$$\text{with } J = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ \star & \ddots & 0 & & \vdots \\ \vdots & \star & 1 & 0 & \vdots \\ \vdots & & \star & \ddots & 0 \\ \star & \dots & \dots & \star & 1 \end{pmatrix}$$

Examples

A realization with the δ-operator is described by :

$$\begin{cases} \delta X_k = A_\delta X_k + B_\delta U_k \\ Y_k = C_\delta X_k + D_\delta U_k \end{cases} \qquad \delta \triangleq \frac{q-1}{\Delta}$$

and it corresponds to the following implicit state-space :

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The Observer State-Feedback

$$\begin{cases} \hat{X}_{k+1} = A_p\hat{X}_k + B_pU_k + K_f(Y_k - C_p\hat{X}_k) \\ U_k = -K_c\hat{X}_k + Q(Y_k - C_p\hat{X}_k) \end{cases}$$

where (A_p, B_p, C_p) corresponds to the plant system and K_c, K_f and Q are the controller's parameters.

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An other possible parametrization

$$\begin{pmatrix} I & 0 & 0 \\ -B_p & I & 0 \\ -I & 0 & I \end{pmatrix} \begin{pmatrix} T_{k+1} \\ \hat{X}_{k+1} \\ U_k \end{pmatrix} = \begin{pmatrix} 0 & -(QC_p + K_c) & Q \\ 0 & (A_p - K_fC) & K_f \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_k \\ \hat{X}_k \\ Y_k \end{pmatrix}$$

Outline

Pole-sensibility stability related measure

The pole-sensitivity measure can be extended in implicit state-space framework.

$$\mu(Z) = \min_{1 \leqslant k \leqslant r} \frac{1 - |\lambda_k|}{\sqrt{N \Psi_k}}$$

whith N represents the number of non trivial elements, and

$$\Psi_k = \left\| \frac{\partial |\lambda_k|}{\partial Z} \times W_Z \right\|_F^2$$

Pole-sensibility stability related measure

Ψ_k can be easily computed :

Proposition 1

$$\frac{\partial |\lambda_k|}{\partial Z} = \bar{M}_1^\top \frac{\partial |\lambda_k|}{\partial \bar{A}} \bar{M}_2^\top$$

with

$$\bar{M}_1 \triangleq \begin{pmatrix} B_p L J^{-1} & 0 & B_p \\ K J^{-1} & I_n & 0 \end{pmatrix}$$

$$\bar{M}_2 \triangleq \begin{pmatrix} J^{-1} N C_p & J^{-1} M \\ 0 & I_n \\ C_p & 0 \end{pmatrix}$$

Pole-sensibility stability related measure

Proposition 2 (Wu, Chen, Li, ... 2001)

Let $(\lambda_k)_{1\leqslant k\leqslant r}$ be the eigenvalue of a matrix $M \in \mathbb{R}^{r\times r}$ and $(x_k)_{1\leqslant k\leqslant r}$ the corresponding right eigenvectors.
Denote $M_x \triangleq (x_1 x_2 \dots x_r)$ and $M_y = (y_1 y_2 \dots y_r) \triangleq M_x^{-H}$.
Then

$$\frac{\partial \lambda_k}{\partial M} = y_k^* x_k^\top \quad \forall k = 1, \dots, r$$

and

$$\frac{\partial |\lambda_k|}{\partial M} = \frac{1}{|\lambda_k|} Re\left(\lambda_k^* \frac{\partial \lambda_k}{\partial M}\right)$$

Outline

Optimal design problem

Denote $\mathscr{R}_H$ the set of equivalent realizations with H as a transfer function.

The *Optimal design problem* consists in finding a realization $\mathcal{R}^{opt}$ in $\mathscr{R}_H$ that maximizes μ

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Optimal design problem

Some subsets of $\mathscr{R}_H$ can be defined from an initial realization Z_0 and a similarity

$$Z = \mathcal{T}_1 Z_0 \mathcal{T}_2$$

with

general case

$$\mathcal{T}_1 = \begin{pmatrix} U & & \\ & T^{-1} & \\ & & I_p \end{pmatrix}, \mathcal{T}_2 = \begin{pmatrix} V & & \\ & T & \\ & & I_m \end{pmatrix}$$

U,V,T non-singular matrices

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δ-operator

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T non-singular matrix

Example

The example used here is a single-input single-output fluid power control system (Njabeleke, Whidborne)

$$A_p = \begin{pmatrix} 9.9988e{-1} & 1.9432e{-5} & 5.9320e{-5} & -6.2286e{-5} \\ -4.9631e{-7} & 2.3577e{-2} & 2.3709e{-5} & 2.3672e{-5} \\ -1.5151e{-3} & 2.3709e{-2} & 2.3751e{-5} & 2.3898e{-5} \\ 1.5908e{-3} & 2.3672e{-2} & 2.3898e{-5} & 2.3667e{-5} \end{pmatrix}$$

$$B_p = \begin{pmatrix} 3.0504e{-3} \\ -1.2373e{-2} \\ -1.2375e{-2} \\ -8.8703e{-2} \end{pmatrix} \quad C_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^\top$$

$$Z_0 = \left(\begin{array}{c|cccc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -3.3071e{-1} & 1 \\ 0 & 1 & 0 & 0 & 1.9869e{+0} & 0 \\ 0 & 0 & 1 & 0 & -3.9816e{+0} & 0 \\ 0 & 0 & 0 & 1 & 3.3255ee{+0} & 0 \\ \hline 0 & -1.6112e{-3} & -1.5998e{-3} & -1.5885e{-3} & -1.5773e{-3} & -8.0843e{-4} \end{array}\right)$$

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Example

For example, it is possible to find the δ-optimal realization

$$Z_{opt}^{\delta} = \underset{det(T)\neq 0}{arg\ max}\ \mu(Z(T))$$

with

$$Z(T) = \begin{pmatrix} T^{-1} & & \\ & T^{-1} & \\ & & I_p \end{pmatrix} Z_0^{\delta} \begin{pmatrix} T & & \\ & T & \\ & & I_m \end{pmatrix}$$

$$Z_{opt}^{\delta} = \left(\begin{array}{cccc|cccc|c} 1 & 0 & 0 & 0 & -4.3728e{+}0 & 2.7770e{+}0 & 1.5953e{+}1 & 2.1160e{+}1 & 3.5644e{-}2 \\ 0 & 1 & 0 & 0 & 2.3090e{+}0 & -1.2959e{+}0 & -6.6800e{+}0 & -9.5796e{+}0 & -2.6145e{-}2 \\ 0 & 0 & 1 & 0 & 6.4736e{+}0 & -4.1528e{+}0 & -2.4059e{+}1 & -3.2103e{+}1 & -1.0745e{-}2 \\ 0 & 0 & 0 & 1 & -1.7320e{+}0 & 1.0786e{+}0 & 6.0998e{+}0 & 8.1425e{+}0 & 1.8563e{-}2 \\ \hline -\Delta & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\Delta & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\Delta & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\Delta & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & -2.8733e{+}0 & 5.6735e{-}1 & -1.3643e{+}0 & 2.7498e{+}0 & -8.0843e{-}4 \end{array}\right)$$

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\hline
-\Delta & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
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Example : some results

realization	$\mu(\mathcal{R})$	parameters
canonical form q	4.4196e-12	9
optimal q	6.8714e-5	25
canonical form δ	1.1699e-5	9
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Conclusions and Perspectives

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- Other structurations to study (q/δ mixed realizations, ...)
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