Réalisations optimales pour l'implantation de systèmes LTI paramétrés

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4 Juillet 2012 — CIFA 2012, Grenoble

Outline

1. Context
2. Classical sensitivity analysis
3. A (new) problem : the parametrized filters
   - Extension to the parametrized filters
4. Resilient implementation of parametrized systems
5. References

Plan

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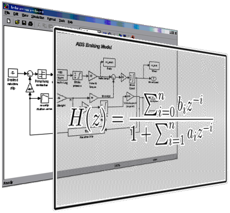

Filters/Controllers
Algorithms

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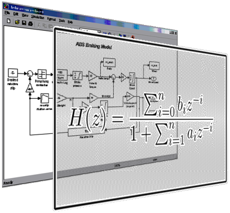

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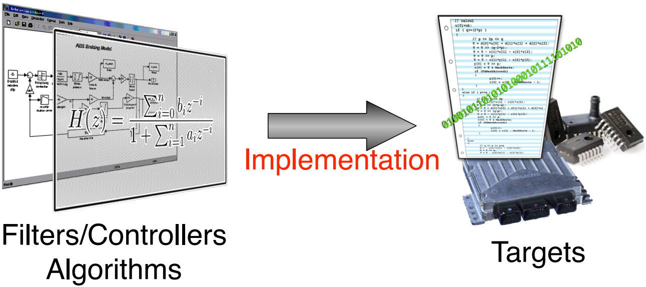
ABS Braking Model
$H(z) = \frac{\sum_{i=0}^{n} b_i z^{-i}}{1+\sum_{i=1}^{n} a_i z^{-i}}$
Filters/Controllers
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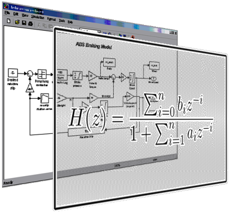

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- Finite precision implementation (fixed-point arithmetic)
- LTI systems
- hardware (FPGA, ASIC) or software (DSP, μC)

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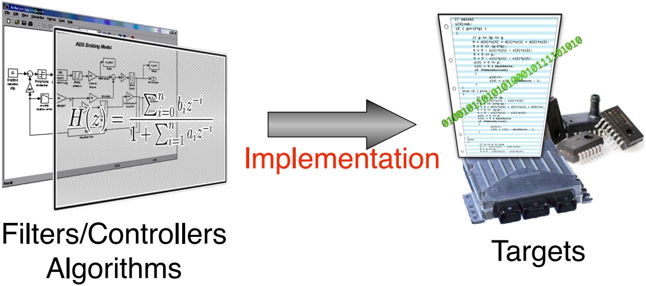

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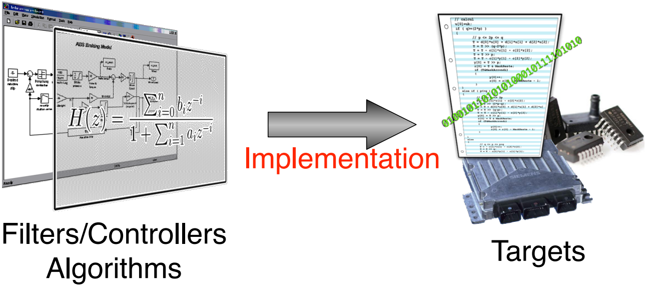

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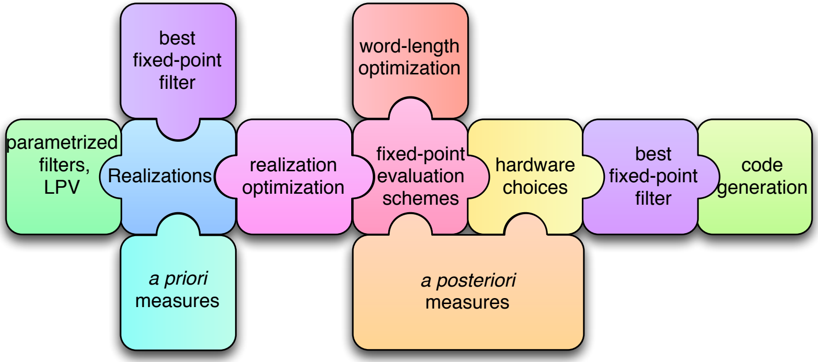
Methodology for the implementation of embedded controllers with finite precision considerations

Global methodology



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L_2-sensitivity

We consider a SISO state-space :

$$\begin{cases} x(k+1) &= \boldsymbol{A}x(k) + \boldsymbol{b}u(k) \\ y(k) &= \boldsymbol{c}x(k) + du(k) \end{cases}$$

Classically, the transfer function sensitivity measure M_{L_2} is used

$$M_{L_2} \triangleq \left\| \frac{\partial h}{\partial \boldsymbol{Z}} \right\|_2^2, \qquad \text{with } \boldsymbol{Z} \triangleq \begin{pmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{c} & d \end{pmatrix}$$

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The associated problem is to find the realization $(\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}, \boldsymbol{T}^{-1}\boldsymbol{b}, \boldsymbol{c}\boldsymbol{T}, d)$ that minimizes the L_2-sensitivity.

Transfer function error – 1

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The approach used is then :

- the fixed-point representation of the coefficients $\boldsymbol{Z}$ changed them in $\boldsymbol{Z}^\dagger = \boldsymbol{Z} + \boldsymbol{\Delta Z}$, where
  - $\boldsymbol{\Delta Z}$ can be considered as independent centered random variables uniformly distributed in range $[-2^{-\gamma_{\boldsymbol{Z}_{ij}}-1}, 2^{-\gamma_{\boldsymbol{Z}_{ij}}-1}]$
  - $\gamma_{\boldsymbol{Z}_{ij}}$ is the length of the fractional part given by

$$\gamma_{\boldsymbol{Z}_{ij}} = \beta_{\boldsymbol{Z}_{ij}} - 2 - \lfloor \log_2 |\boldsymbol{Z}_{ij}| \rfloor$$

  - and $\beta_{\boldsymbol{Z}_{ij}}$ is the word-length used to represent $\boldsymbol{Z}_{ij}$.

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  - and $\beta_{\boldsymbol{Z}_{ij}}$ is the word-length used to represent $\boldsymbol{Z}_{ij}$.
- the coefficient's quantification changes the transfer function h in $h^\dagger = h + \Delta h$

Transfer function error – 2

Δh is a transfer function with random variables as coefficients.
So, a measure of the transfer function error can statistically be defined by [2]

$$\sigma_{\Delta h}^2 \triangleq \frac{1}{2\pi} \int_0^{2\pi} E\left\{ \left\| \Delta h\left(e^{j\omega}\right) \right\|_F^2 \right\} \, d\omega$$

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and can be computed by [1]

$$\sigma^2_{\Delta h} = \left\| \frac{\partial h}{\partial Z} \times \Xi_Z \right\|_2^2$$

with

$$\left(\Xi_Z\right)_{ij} \triangleq \begin{cases} \frac{2^{-\beta_{Z_{ij}}+1}}{\sqrt{3}} \lfloor Z_{ij} \rfloor_2 & \text{if } Z_{ij} \text{ non-trivial} \\ 0 & \text{if } Z_{ij} \text{ is trivial} \end{cases}$$

and $\lfloor x \rfloor_2$ is nearest power of 2 lower than $|x|$

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Parametrized filters : an extended problem

We now consider LTI filters/controllers where the coefficients depend on extra parameters $\boldsymbol{\theta}$.

- These parameters are fixed, but unknown at implementation and compile-time ;
- We only know intervals they belong to ;
- The coefficients are computed once *in-situ* at initialization-time ;

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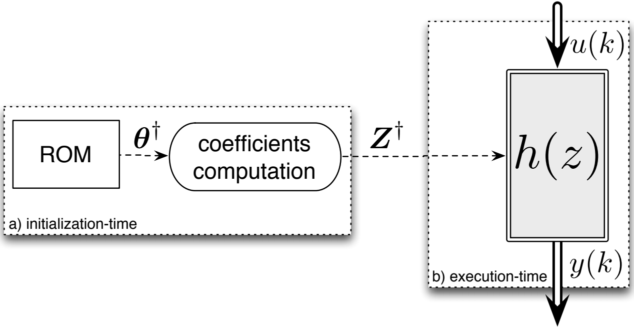
☞this is widely used by car manufacturers in order to calibrate controllers much more later in the development lifecycle.

Parametrized filters



Initialization and execution

Example – 1

Continuous-time 2nd order filter

$$h(s) = \frac{g}{s^2 + 2\xi\omega_c s + \omega_c^2} \quad .$$

Depends on 3 parameters :

- g gain (g/ω_c is the static gain) ;
- ξ quality factor ;
- ω_c cutoff frequency.

Example – 2

Discrete-time 2nd order filter

$$H(z) = \frac{b_0 z^2 + b_1 z + b_2}{a_0 z^2 + a_1 z + a_2}$$

with

- $b_0 = gT^2, \quad b_1 = 2gT^2, \quad b_2 = gT^2$
- $a_0 = 4\xi\omega_c T + \omega_c^2 T^2 + 4$, $a_1 = 2\omega_c^2 T^2 - 8$, $a_2 = \omega_c^2 T^2 - 4\xi\omega_c T + 4$

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The extra parameters are $\boldsymbol{\theta} = \begin{pmatrix} g \\ T \\ \omega_c \\ \xi \end{pmatrix}$ or $\boldsymbol{\theta} = \begin{pmatrix} g \\ Fe \\ F_c \\ \pi \\ \xi \end{pmatrix}, \ldots$

and if a direct form is used, we have the relationship between the coefficients and the parameters.

Example – 3

It can also be implemented with a controllable canonical form

$$\begin{cases} x(k+1) &= \begin{pmatrix} -\frac{a_1}{a_0} & -\frac{a_2}{a_0} \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) \\ y(k) &= \begin{pmatrix} \frac{b_1 a_0 - b_0 a_1}{a_0^2} & \frac{b_2 a_0 - b_0 a_2}{a_0^2} \end{pmatrix} x(k) + \frac{b_0}{a_0} u(k) \end{cases}$$

Or any other state-space realization.

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Or any other state-space realization.
Or any other *interesting* form (ρDFIIt, etc.).
☞In that cases, the computation of the coefficients (from the parameters) can be much more complicated...
But it is only done once, at initialization-time.

Quantization of the parameters – 1

Questions :

- What will be the impact of the quantization of these parameters ?
- How the transfer function and/or poles will be affected ?
- How to consider them for the search for *optimal* realization ?

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- What will be the impact of the quantization of these parameters ?
- How the transfer function and/or poles will be affected ?
- How to consider them for the search for *optimal* realization ?
  - consider the worst degradation among the set of parameters, and optimize it ?
  - search for the parameters that gives the worst case ?
  - ...

Quantization of the parameters – 2

So, we have considered the following case :

- the parameters $\boldsymbol{\theta}$ can be quantized (in $\boldsymbol{\theta}^\dagger$) (some of them will not moved because they can be exactly represented)

Quantization of the parameters – 2

So, we have considered the following case :

- the parameters $\boldsymbol{\theta}$ can be quantized (in $\boldsymbol{\theta}^\dagger$) (some of them will not moved because they can be exactly represented)
- the coefficients $\boldsymbol{Z}(\boldsymbol{\theta}^\dagger)$ are computed with enough precision, and then are quantized in $\boldsymbol{Z}^\dagger$
  $\rightarrow$ similar than suppose that a quantization will occur on the exact computation of $\boldsymbol{Z}$, with the quantized parameters $\boldsymbol{\theta}^\dagger$

Quantization of the parameters – 3

In a similar way :

- $\boldsymbol{\theta}$ quantized in $\boldsymbol{\theta}^\dagger = \boldsymbol{\theta} + \boldsymbol{\Delta\theta}$
  since $\boldsymbol{\theta}$ is unknown, we can only say $\boldsymbol{\Delta\theta}_k$ are independent centered random variables, uniformly distributed in range $[-2^{-\gamma_{\boldsymbol{\Delta\theta}_k}-1}, 2^{-\gamma_{\boldsymbol{\Delta\theta}_k}-1}]$

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- for Z, we will compute $Z(\theta^\dagger)$ and then quantized it in $Z^\dagger = Z(\theta^\dagger) + \Delta Z$ with $\sigma^2_{\Delta Z_{ij}} = \frac{2^{-2\gamma_{Z_{i,j}}}}{12}\delta_{Z_{ij}}$

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A first order approximation gives :

$$\boldsymbol{Z} - \boldsymbol{Z}^\dagger \approx \sum_k \frac{\partial \boldsymbol{Z}}{\partial \boldsymbol{\theta}_k} \boldsymbol{\Delta\theta}_k + \boldsymbol{\Delta Z}$$

Extension of the transfer function error measure

In that context, the transfer function error measure can be extended :

$$\sigma^2_{\Delta h} = \left\| \frac{\partial H}{\partial Z} \times \boldsymbol{\Xi}_Z \right\|_2^2 + \sum_k \left\| \frac{\partial h}{\partial Z} \times \boldsymbol{\Theta}_k \right\|_2^2$$

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$\rightarrow$ quantization of Z

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→ quantization of each $\boldsymbol{\theta_k}$

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- $\frac{\partial Z}{\partial \theta_k}$ can be obtained via symbolic computations
- ϟ the quantization of Z is supposed to be *best roundoff*. For *truncature*, the expression is no more valid !

Pole error measure

The same reasoning can be applied to poles λ_l (or their moduli). Let us define

$$\sigma^2_{\Delta|\lambda|} \triangleq \sum_i \omega_i E\left\{\left(\Delta\left|\lambda_i\right|\right)^2\right\}$$

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$\rightarrow$ analytical expression for $\frac{\partial |\lambda_i|}{\partial Z}$

Example – 4

For the previous example (canonical state-space)

$$\boldsymbol{Z} = \begin{pmatrix} -\frac{2\left(T^2\omega_c^2-4\right)}{T^2\omega_c^2+4T\omega_c\xi+4} & -\frac{T^2\omega_c^2-4T\omega_c\xi+4}{T^2\omega_c^2+4T\omega_c\xi+4} & 1 \\ 1 & 0 & 0 \\ 2T^2g - \frac{2\left(T^2\omega_c^2-4\right)T^2g}{(T^2\omega_c^2+4T\omega_c\xi+4)^2} & T^2g - \frac{2\left(T^2\omega_c^2-4\right)T^2g}{(T^2\omega_c^2+4T\omega_c\xi+4)^2} & \frac{T^2g}{T^2\omega_c^2+4T\omega_c\xi+4} \end{pmatrix}$$

And symbolic derivatives gives

$$\frac{\partial \boldsymbol{Z}}{\partial g} = \begin{pmatrix} \frac{8(T\omega_c-2)(T\omega_c+2)T\omega_c}{(T^2\omega_c^2+4T\omega_c\xi+4)^2} & \frac{8\left(T^2\omega_c^2+4\right)T\omega_c}{(T^2\omega_c^2+4T\omega_c\xi+4)^2} & 0 \\ 0 & 0 & 0 \\ \frac{16(T\omega_c-2)(T\omega_c+2)T^3g\omega_c}{(T^2\omega_c^2+4T\omega_c\xi+4)^3} & \frac{16(T\omega_c-2)(T\omega_c+2)T^3g\omega_c}{(T^2\omega_c^2+4T\omega_c\xi+4)^3} & \frac{-4T^3g\omega_c}{(T^2\omega_c^2+4T\omega_c\xi+4)^2} \end{pmatrix}$$

and similar results for $\frac{\partial \boldsymbol{Z}}{\partial \xi}$, $\frac{\partial \boldsymbol{Z}}{\partial T}$ and $\frac{\partial \boldsymbol{Z}}{\partial \omega_c}$.

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Optimal resilient implementation – 1

Let denotes $\mathcal{R}_{\boldsymbol{Z}_0}$ the set of equivalent state-space realizations of a given realization $\boldsymbol{Z}_0$.
It is of interest to consider the optimal problem

$$\underset{\boldsymbol{Z}\in\mathcal{R}_{\boldsymbol{Z}_0}}{\arg\min}\ \sigma^2_{\Delta h}$$

i.e. find the realization that minimizes the impact of the quantization of $\boldsymbol{Z}$ and $\boldsymbol{\theta}$

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i.e. find the realization that minimizes the impact of the quantization of $\boldsymbol{Z}$ and $\boldsymbol{\theta}$

$$\boldsymbol{Z}(\boldsymbol{T}) = \begin{pmatrix} \boldsymbol{T}^{-1} & \\ & 1 \end{pmatrix} \boldsymbol{Z}_0 \begin{pmatrix} \boldsymbol{T} & \\ & 1 \end{pmatrix}$$

$$\boldsymbol{T}_{opt} = \arg\min_{\mathcal{T} \text{ invertible}} \sigma^2_{\Delta h}(\boldsymbol{T}).$$

☞A global optimization algorithm (Adaptative Simulated Annealing) is used.

Optimal resilient implementation – 2

Obviously, the previous slide is for a given $\boldsymbol{\theta} = \boldsymbol{\theta}_0$.
It is much more interesting to consider a set of possible values for $\boldsymbol{\theta}$, denoted $\boldsymbol{\Omega_\theta}$ (a box if the parameters are defined in intervals).

Optimal resilient implementation – 2

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It is much more interesting to consider a set of possible values for $\boldsymbol{\theta}$, denoted $\boldsymbol{\Omega}_{\boldsymbol{\theta}}$ (a box if the parameters are defined in intervals).

- A worst-case measure can be built (for state-space, depends on the transformation matrix $\boldsymbol{\mathcal{U}}$)

$$\sigma^2_{\Delta h, \boldsymbol{\Omega}_{\boldsymbol{\theta}}}(\boldsymbol{T}) \triangleq \max_{\boldsymbol{\theta} \in \boldsymbol{\Omega}_{\boldsymbol{\theta}}} \sigma^2_{\Delta h}(\boldsymbol{\theta}, \boldsymbol{T})$$

  And the optimal resilient implementation problem is

$$\boldsymbol{T}_{opt} = \underset{\boldsymbol{T} \text{ inversible}}{\arg\ \min}\ \sigma^2_{\Delta h, \boldsymbol{\Omega}_{\boldsymbol{\theta}}}(\boldsymbol{T})$$

- Car manufacturers are interested in finding $\boldsymbol{\theta}$ that gives a maximal sensitivity *for a given realization*

$$\underset{\boldsymbol{\theta} \in \Omega_{\boldsymbol{\theta}}}{\arg\ \max}\ \sigma^2_{\Delta h}(\boldsymbol{\theta})$$

  in order to focus their effort on that particular configuration

Optimal resilient implementation – 3

We do not have yet a real solution to propose.

- If the number of parameters is very low, it is possible to use a grid approximation of $\boldsymbol{\Omega_\theta}$ (*i.e.* discretize each interval $[\underline{\theta_k};\overline{\theta_k}]$ in n points, and do exhaustive search on these n^a $\boldsymbol{\theta}$)

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- We are also looking at interval arithmetic (intervals, affine arithmetics, Taylor approximation, etc.)

Conclusion

- The parametrized filter implementation problem has been defined
- Some (extended) transfer function error measures can be used to deal with the impact of the quantized parameters
- It has been extended to the *Specialized Implicit Framework*
- But still some work to do for the optimal problem when the parameters belong to a set

Any questions ?

Plan

1 Context

2 Classical sensitivity analysis

3 A (new) problem : the parametrized filters
   Extension to the parametrized filters

4 Resilient implementation of parametrized systems

5 References

References

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