Sum-of-products Evaluation Schemes with Fixed-Point arithmetic, and their application to IIR filter implementation

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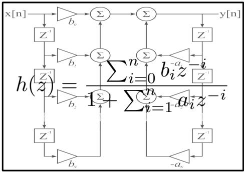
Outline

On the first hand... A filter



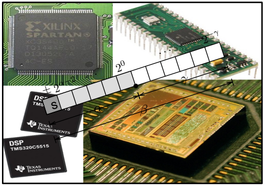
- Signal Processing
- LTI filters: FIR or IIR
- Its transfer function:

$$h(z) = \frac{\sum_{i=0}^{n} b_i z^{-i}}{1 + \sum_{i=1}^{n} a_i z^{-i}}$$

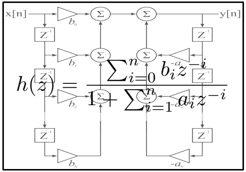
- Algorithmic relationship used to compute output(s) from input(s), for example:

$$y(k) = \sum_{i=0}^{n} b_i u(k-i) - \sum_{i=1}^{n} a_i y(k-i)$$

On the other hand... A target



- Hardware target (FPGA, ASIC) or software target (DSP, μC)
- Using fixed-point arithmetic for different reasons:
  - no FPU
  - cost
  - size
  - power consumption
  - etc.



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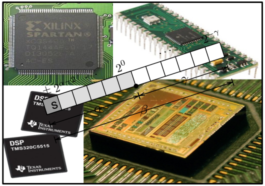
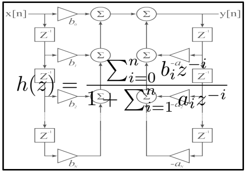


Need

Methodology and tools for the implementation of embedded filter algorithms in fixed-point arithmetic.



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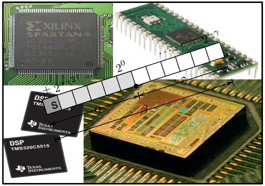

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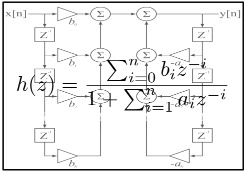
A first (and basic) methodology

1. Given a filter, choose an algorithm
2. Round the coefficients in fixed-point arithmetic
3. Implement algorithm



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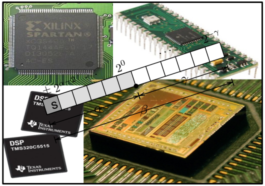


Need

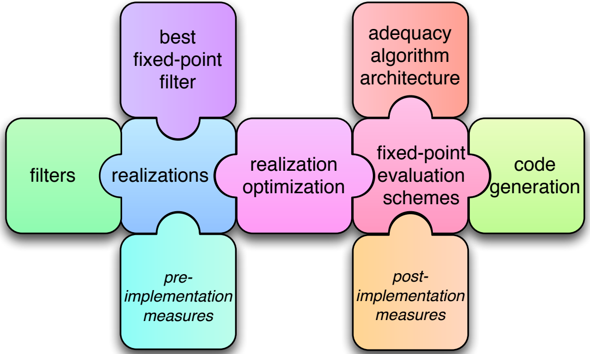
Methodology and tools for the implementation of embedded filter algorithms in fixed-point arithmetic.

A first (and basic) methodology

1. Given a filter, choose an algorithm
   - There are many possible realizations
2. Round the coefficients in fixed-point arithmetic
   - What format? Depends on the choice of algorithm
3. Implement algorithm
   - Is there only one possible implementation?

From filter to code

This work is a part of a global approach, transforming filters into fixed-point codes.



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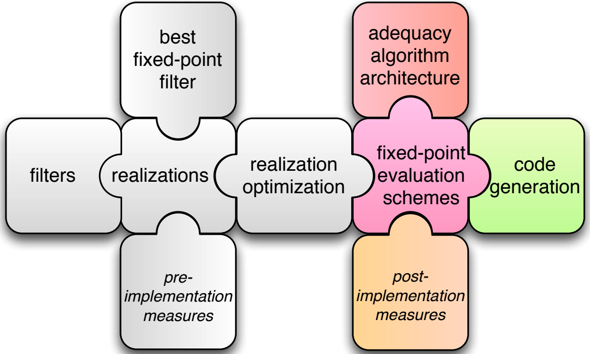

Objective:

Given an algorithm and a target, find the *optimal* implementation.

- model the fixed-point algorithms
- model the hardware resources (computational units, etc.)

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Given an algorithm and a target, find the *optimal* implementation.

- model the fixed-point algorithms
- model the hardware resources (computational units, etc.)
- evaluate the degradation
- find one/some *optimal* implemented algorithm(s)

Outline

Evaluation Scheme

The only operations needed in filter algorithm computation are sum-of-products:

$$S = \sum_{i=1}^{n} c_i \cdot x_i$$

where c_i are known constants and x_i variables (inputs, state or intermediate variables).

Evaluation Scheme

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$$S = \sum_{i=1}^{n} c_i \cdot x_i$$

where c_i are known constants and x_i variables (inputs, state or intermediate variables).

Question:

How to implement S in fixed-point arithmetic?

SoP Example

Let H be the transfer function of a butterworth filter of $4^{th}-$order:

$$H(z) = \frac{0.00254078 + 0.01016312z^{-1} + 0.01524469z^{-2} + 0.01016312z^{-3} + 0.00254078z^{-4}}{1 - 2.64402372z^{-1} + 2.77901148z^{-2} - 1.34558515z^{-3} + 0.25124989z^{-4}}$$

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Associated algorithm (Direct Form 1):

$$\begin{aligned} y(k) \quad = \quad & 0.00254078\ u(k) + 0.01016312z\ u(k-1) + 0.01524469z\ u(k-2) \\ & +0.01016312z\ u(k-3) + 0.00254078\ u(k-4) + 2.64402372\ y(k-1) \\ & -2.77901148\ y(k-2) + 1.34558515\ y(k-3) - 0.25124989\ y(k-4) \end{aligned}$$

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Associated algorithm (integer part on 16 bits):

$$\begin{aligned} y(k) \quad = \quad & 21313.60743424 \times 2^{-23} \times u(k) + 21313.60743424 \times 2^{-21} \times u(k-1) \\ & +31970.43212288 \times 2^{-21} \times u(k-2) + 21313.60743424 \times 2^{-21} \times u(k-3) \\ & +21313.60743424 \times 2^{-23} \times u(k-4) + 21659.84231424 \times 2^{-13} \times y(k-1) \\ & -22765.66204416 \times 2^{-13} \times y(k-2) + 22046.0670976 \times 2^{-14} \times y(k-3) \\ & -16465.91279104 \times 2^{-16} \times y(k-4) \end{aligned}$$

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Associated algorithm (round to nearest integer):

$$\begin{aligned} y(k) \quad = \quad & 21314 \times 2^{-23} \times u(k) + 21314 \times 2^{-21} \times u(k-1) \\ & + 31970 \times 2^{-21} \times u(k-2) + 21314 \times 2^{-21} \times u(k-3) \\ & + 21314 \times 2^{-23} \times u(k-4) + 21660 \times 2^{-13} \times y(k-1) \\ & - 22766 \times 2^{-13} \times y(k-2) + 22046 \times 2^{-14} \times y(k-3) \\ & - 16466 \times 2^{-16} \times y(k-4) \end{aligned}$$

In software, addition is commutative but maybe not associative

On 4 bits

$$\begin{array}{ccc} 0.5 & =_2 & 0_\bullet 100 \\ 1.5 & =_2 & 01_\bullet 10 \\ 4 & =_2 & 0100_\bullet \end{array}$$

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$4 + (0.5 + 1.5) =_2 010_\bullet 0 + 0100_\bullet =_2 0110_\bullet =_{10} 6$

$(4 + 0.5) + 1.5 =_2 0100_\bullet + 01_\bullet 10 =_2 0101_\bullet =_{10} 5$

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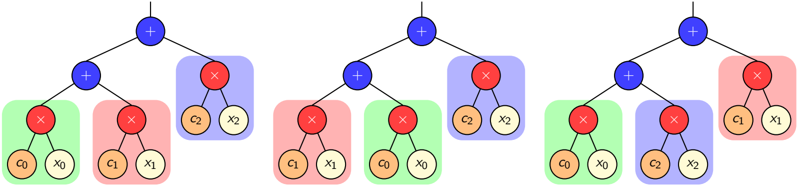
Consider the order is important.



(a) (b) (c)

(a) = (b) but (a) $\neq$ (c)

oSoP

An evaluation scheme for a given SoP with a given order will be called ordered-Sum-of-Products (oSoP).

Number of oSoPs

For a given SoP of $N^{th}-$order, there are $\prod_{i=1}^{N-1}(2i-1)$ possible oSoPs to consider.

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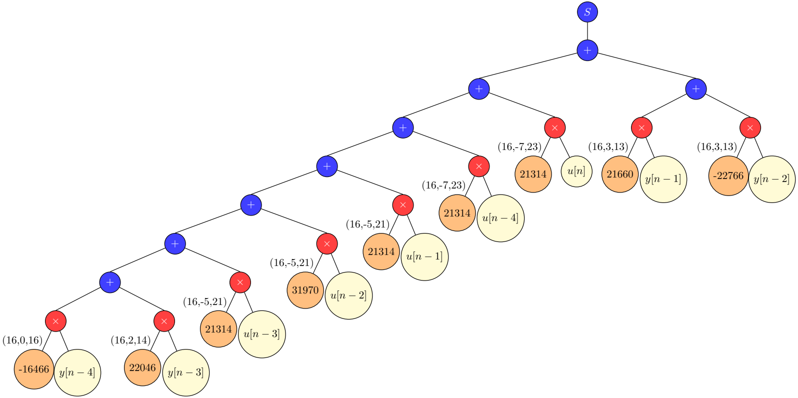
Associated algorithm:

$$\begin{aligned} y(k) \quad = \quad & 21314 \times 2^{-23} \times u(k) + 21314 \times 2^{-21} \times u(k-1) \\ & +31970 \times 2^{-21} \times u(k-2) + 21314 \times 2^{-21} \times u(k-3) \\ & +21314 \times 2^{-23} \times u(k-4) + 21660 \times 2^{-13} \times y(k-1) \\ & -22766 \times 2^{-13} \times y(k-2) + 22046 \times 2^{-14} \times y(k-3) \\ & -16466 \times 2^{-16} \times y(k-4) \end{aligned}$$

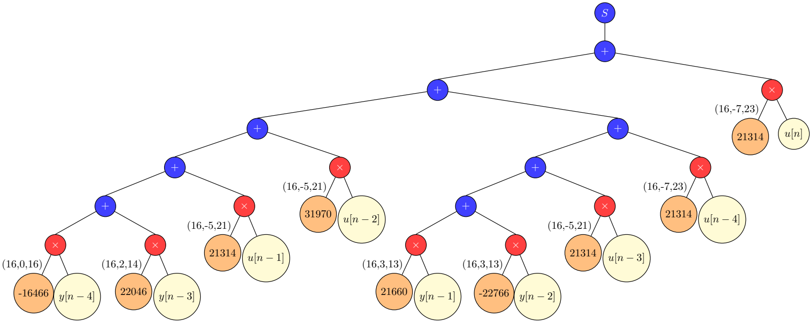
There are here 9 multiplications, so we have $\prod_{i=1}^{8}(2i-1) \simeq 2$ millions oSoPs to consider.

For example...

oSoP Example 1



oSoP Example 2



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Question: Which oSoP should we choose?

To do this, we developed a tool that:

- generates all possible oSoPs
- propagates fixed-point representation
- evaluates degradation errors

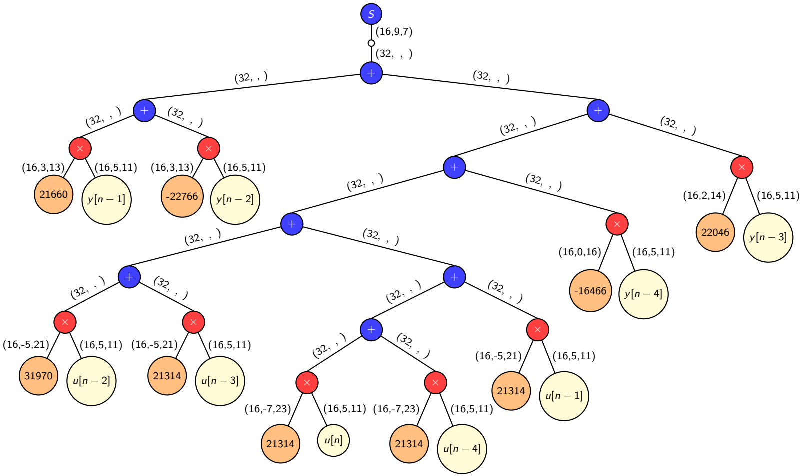
Fixed-Point propagation

FPR

Fixed-Point Representation (FPR) is defined as the tuple (wordlength, integer part, fractional part).

What are the given (input) values?

- FPR of constants c_i's (given by wordlength and value of c_i)
- FPR of variables x_i's
- Wordlength of adders and multipliers
- Final FPR

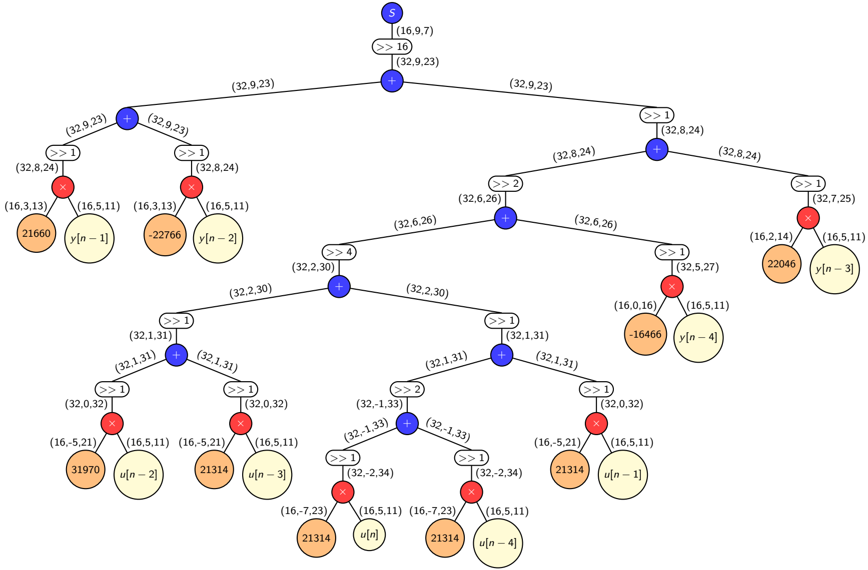

S
(16,9,7)
(32, ,)
21660
y[n − 1]
-22766
y[n − 2]
22046
y[n − 3]
-16466
y[n − 4]
31970
u[n − 2]
21314
u[n − 3]
21314
u[n]
21314
u[n − 4]
21314
u[n − 1]
(16,3,13)
(16,5,11)
(16,2,14)
(16,0,16)
(16,-5,21)
(16,-7,23)

Propagation Rules

From an oSoP parametrized with inputs FPR and wordlength, and using some propagation rules on adders and multipliers, we obtain a fully-parametrized oSoP.

Form the previous oSoP, we have the following fully-parametrized oSoP...

Propagation



Fixed-Point computational errors

Question:

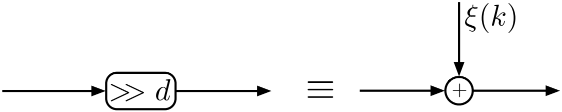
How to evaluate the numerical degradations?

Fixed-Point implementation implies numerical degradations, which depend on:

- the way the computations are organized
- the fixed-point representation of all the signals used in the computations
- and the fixed-point representation of each step of the operations

Noise

Usually in signal processing, we see degradation errors like additive white uniformly distributed noises.

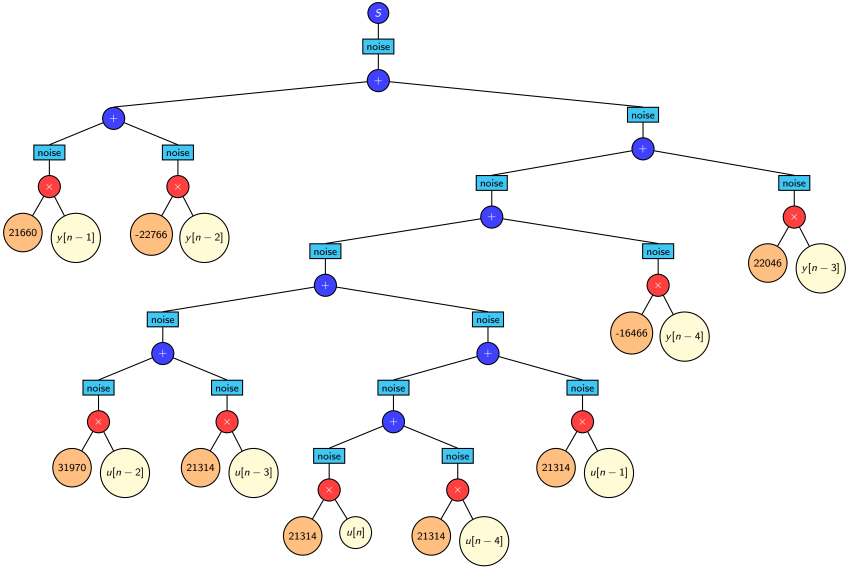


Right-shift of d bits:

For a right-shift of d bits, first (μ) and second (σ) order moment are:

	Truncation	Best roundoff
μ	$2^{-\gamma-1}(1-2^{-d})$	$2^{-\gamma-d-1}$
σ^2	$\frac{2^{-2\gamma}}{12}(1-2^{-2d})$	$\frac{2^{-2\gamma}}{12}(1-2^{-2d})$

Degradation errors



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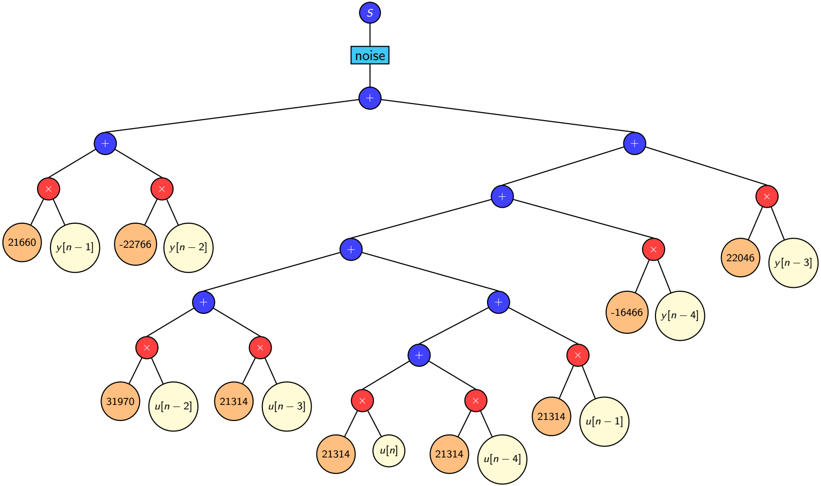

Noise

- Different noises are added through the tree
- This cumulated noise can be viewed as output of a filter

$$y(k) = \sum_{i=0}^{n} b_i u(k-i) - \sum_{i=1}^{n} a_i y(k-i)$$

$$y^{\dagger}(k) = \sum_{i=0}^{n} b_i u(k-i) - \sum_{i=1}^{n} a_i y^{\dagger}(k-i) + \xi(k)$$

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$$e(k) \triangleq y^{\dagger}(k) - y(k) = \xi(k) - \sum_{i=1}^{n} a_i e(k-i)$$

$$h_{\xi}(z) = \frac{1}{1 + \sum_{i=1}^{n} a_i z^{-i}}$$

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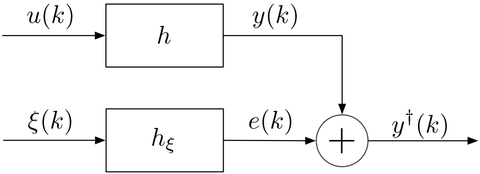

oSoP choice

Once we have evaluated degradations through our oSoPs, we want to choose the *best* one.

There are different criteria to choose oSoPs, like:

- noise
    - couple mean/variance

oSoP choice

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There are different criteria to choose oSoPs, like:

- noise
- latency (infinite parallelism)
  - height of the syntax tree
  - depending on the number of operators of the target

oSoP choice

Once we have evaluated degradations through our oSoPs, we want to choose the *best* one.

There are different criteria to choose oSoPs, like:

- noise
- latency (infinite parallelism)
- adequacy with hardware target
  - number of operators
  - wordlength of operators
  - *etc.*

oSoP choice

Once we have evaluated degradations through our oSoPs, we want to choose the *best* one.

There are different criteria to choose oSoPs, like:

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- latency (infinite parallelism)
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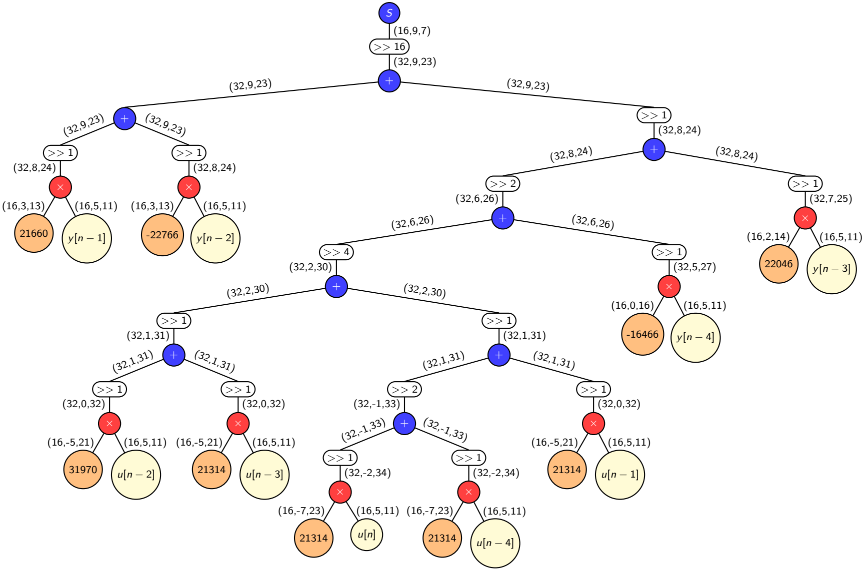
- *etc.*

Some options

Roundoff around multiplication

- Roundoff After Multiplication (RAM)


S
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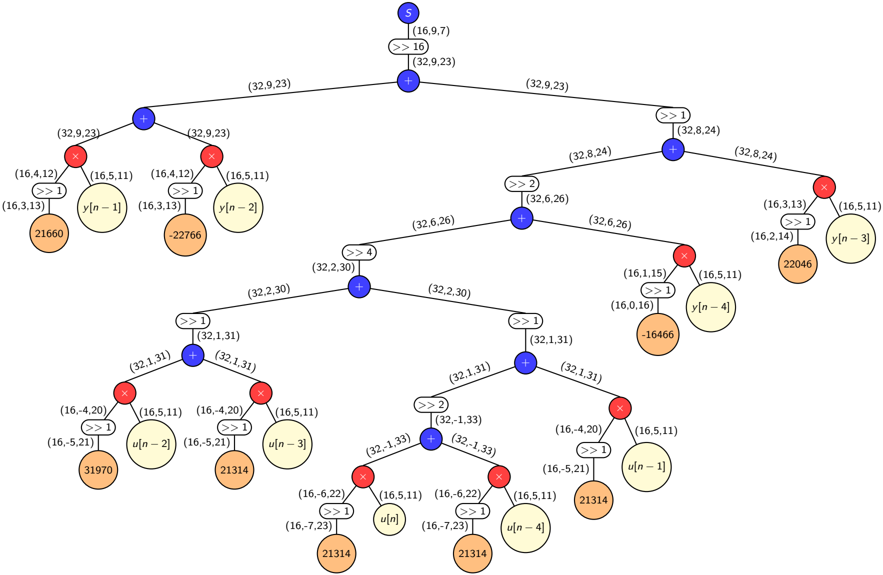
Some options

Roundoff around multiplication

- Roundoff After Multiplication (RAM)
- Roundoff Before Multiplication (RBM)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u[n − 4]
21314

Some options

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Without shift

Target may oblige us to have no shifts. So, all shifts needed are deferred to constants.

Outline

Conclusion

We try to answer the following question:

For a given sum-of-products, how to produce *optimal* implementation?

The main results are a tool and a methodology:

- Model the various evaluation schemes
- Propagate FPR through trees
- Evaluate degradations

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The main results are a tool and a methodology:

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A lot of work still to be done:

- Propagate intervals rather than FPR
- Consider adequacy with hardware resources
- Consider a more realistic model for degradation errors
- Release the source code of our tool

Thank You

Any Questions?