Formatting bits to better implement signal processing algorithms

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Outline

In PECCS'14

Smartphone applications, measurement applications, embedded devices, ...

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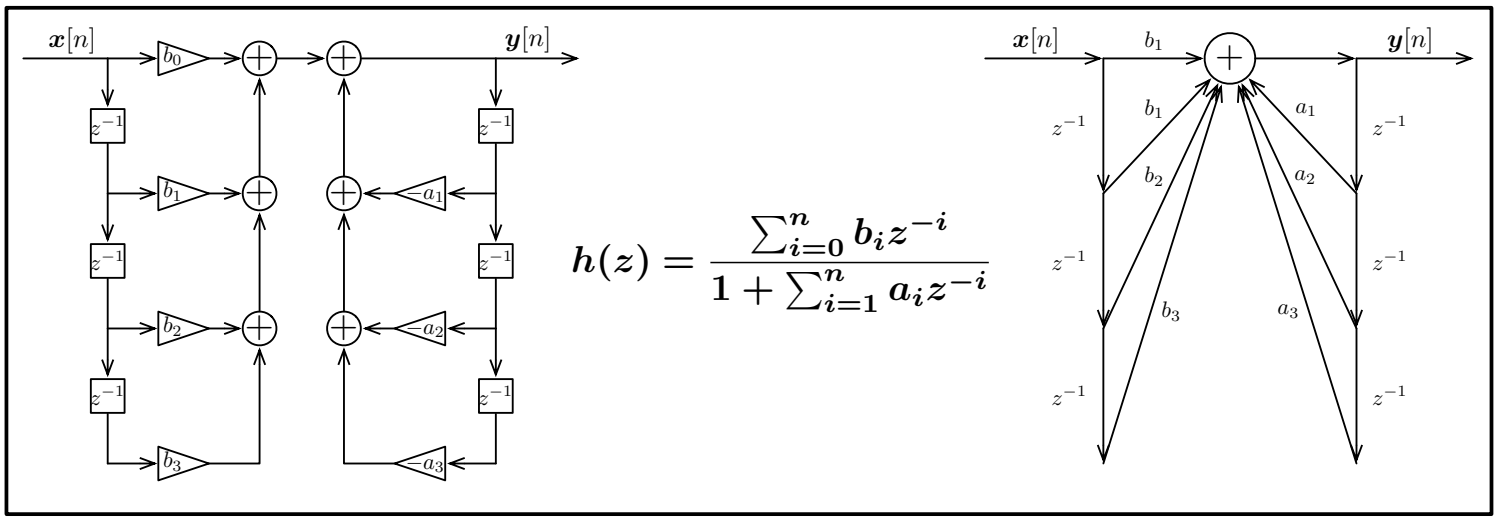
Implementation problem

We need methodology and tools for the implementation of embedded filter algorithms with only integer arithmetic.

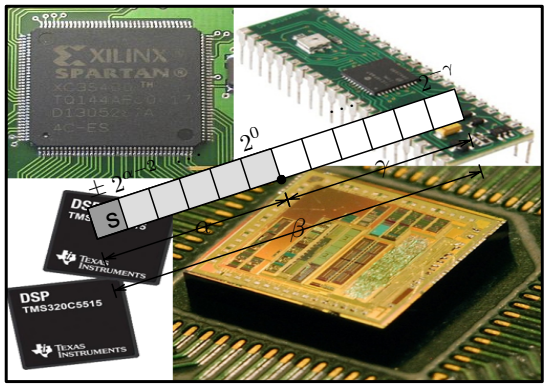
On the first hand... A filter



- Signal Processing
- LTI filters: FIR or IIR
- Its transfer function
- Algorithmic relationship used to compute output(s) from input(s), for example:

$$y(k) = \sum_{i=0}^{n} b_i u(k-i) - \sum_{i=1}^{n} a_i y(k-i)$$

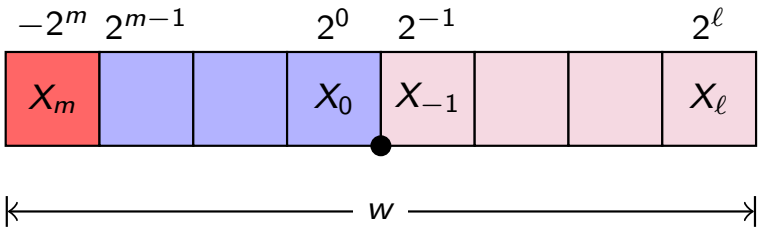
On the other hand... A target



- Hardware target (FPGA, ASIC) or software target (DSP,μC)
- Due to resources constraints (cost, size, power consumption, ...) we have no FPU, so we can only use fixed-point arithmetic

Fixed-Point number

Fixed-point numbers are integers used to approximate real numbers.



- Representation : $X.2^\ell$ with $X = X_m X_{m-1} ... X_0 ... X_\ell$.
- Format : determined by wordlength and fixed-point position, and noted for example (m, ℓ).

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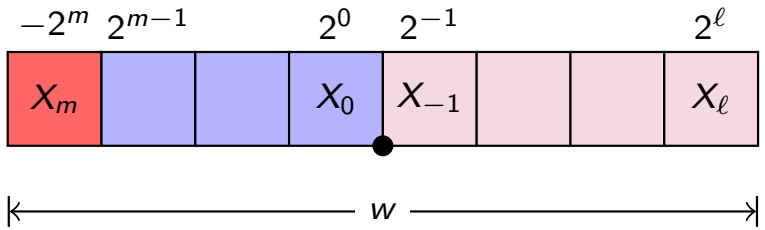

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Computation in finite precision and choice of formats imply errors.

Numerical degradations

- quantization of the coefficients
- round-off errors in computations

IIR Filter

Let H be the transfer function of a $n-th$ order IIR filter :

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_n z^{-n}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}, \quad \forall z \in \mathbb{C}. \tag{1}$$

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This filter is usually realized with the following algorithm

$$y(k) = \sum_{i=0}^{n} b_i u(k-i) - \sum_{i=1}^{n} a_i y(k-i) \tag{2}$$

where $u(k)$ is the input at step k and $y(k)$ the output at step k.

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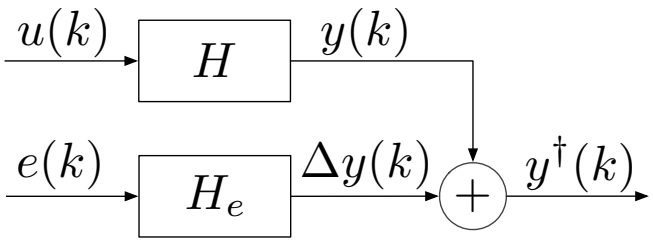
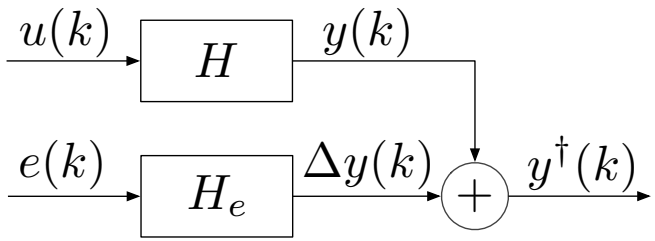
where $u(k)$ is the input at step k and $y(k)$ the output at step k. We can see round-off errors as the add of an error $e(k)$ on the output and only $y^\dagger(k)$ can be computed.

$$y^\dagger(k) = \sum_{i=0}^{n} b_i u(k-i) - \sum_{i=1}^{n} a_i y^\dagger(k-i) + e(k). \tag{3}$$



$\Delta y(k) \triangleq y^\dagger(k) - y(k)$ can be seen as the result of the error through the filter H_e :

$$H_e(z) = \frac{1}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}, \quad \forall z \in \mathbb{C}.$$



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If the error $e(k)$ is in $[\underline{e}; \overline{e}]$, then we are able to compute $\underline{\Delta y}$ and $\overline{\Delta y}$ such that $\Delta y(k)$ is in $[\underline{\Delta y}; \overline{\Delta y}]$:

$$\begin{aligned} \underline{\Delta y} &= \frac{\overline{e} + \underline{e}}{2} |H_e|_{DC} - \frac{\overline{e} - \underline{e}}{2} \|H_e\|_{\ell\infty} \\ \overline{\Delta y} &= \frac{\overline{e} + \underline{e}}{2} |H_e|_{DC} + \frac{\overline{e} - \underline{e}}{2} \|H_e\|_{\ell\infty} \end{aligned}$$

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Objective:

Given an algorithm and a bound on the final error, find an implementation which reduces the number of bits of the computation while controlling the error on the output result.

Outline

Sum-of-Products

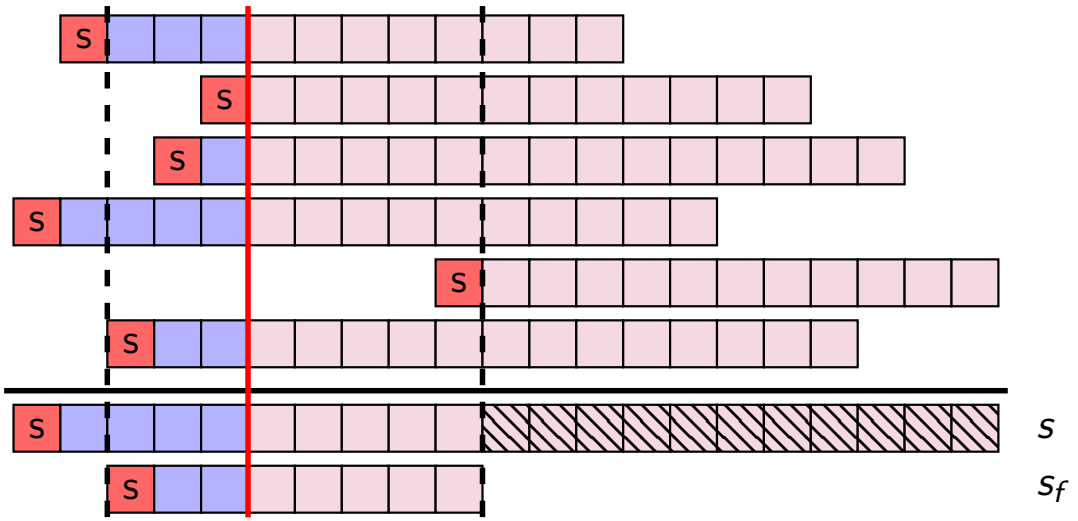
The only operations needed in filter algorithm computation are sum-of-products:

$$s = \sum_{i=1}^{n} c_i \cdot x_i = \sum_{i=1}^{n} p_i$$

where c_i are known constants and x_i variables (inputs, state or intermediate variables).

In the context of filter design, **we know the fixed-point format of the final result.**

Formatting



Context

A sum of N terms $(p_i)_{1 \leq i \leq N}$ with different formats, and the known FPF of final result (s_f), less than total wordlength (s).

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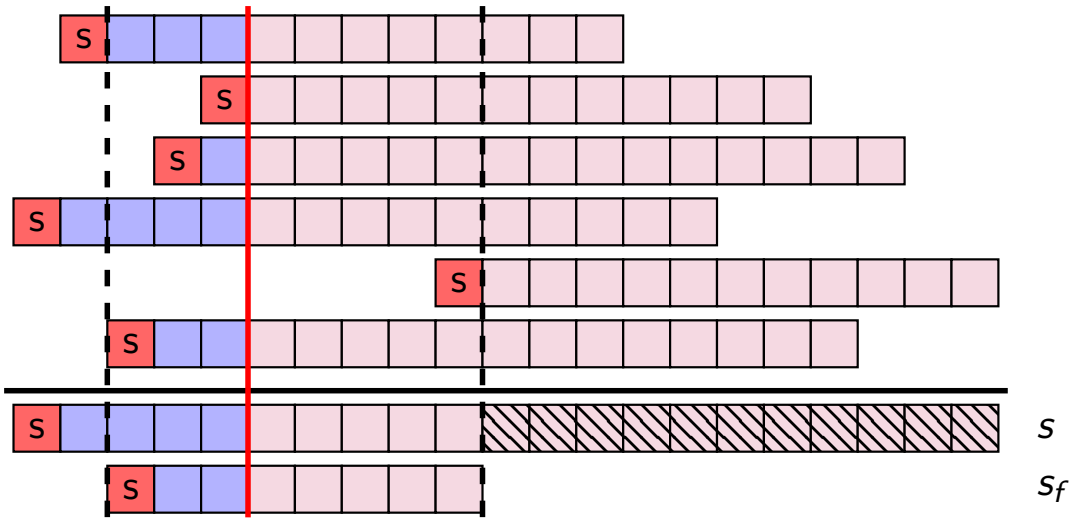

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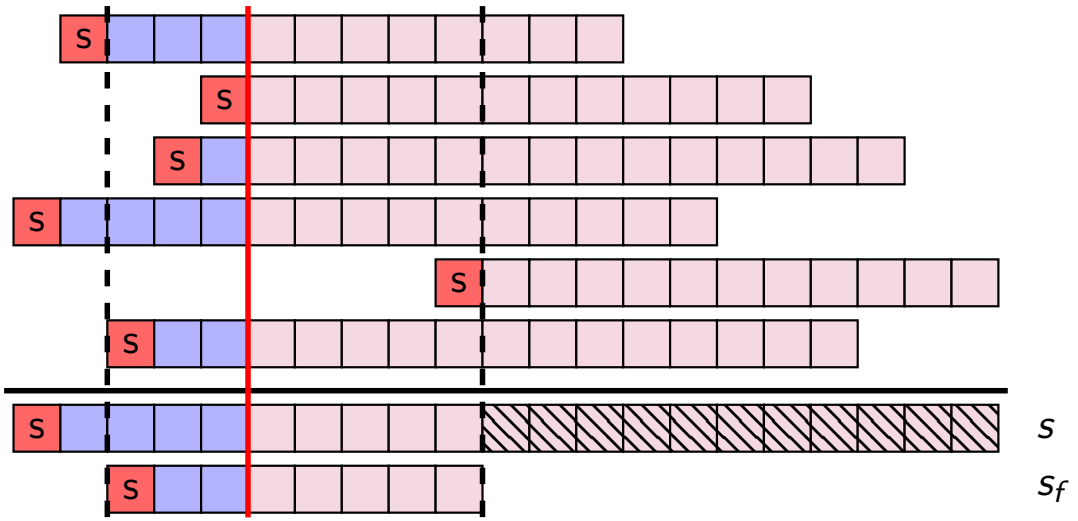
Questions:

Can we remove bits that don't impact the final result ? If we want a faithful round-off of the final result, can we remove some bits ?

Formatting



Two-step formatting

1. most significant bits
2. least significant bits

Jacskon's Rule (1979)

This Rule states that in consecutive additions and/or subtractions in **two's complement arithmetic**, some intermediate results and operands may overflow. As long as the final result representation can handle the final result without overflow, then the result is valid.

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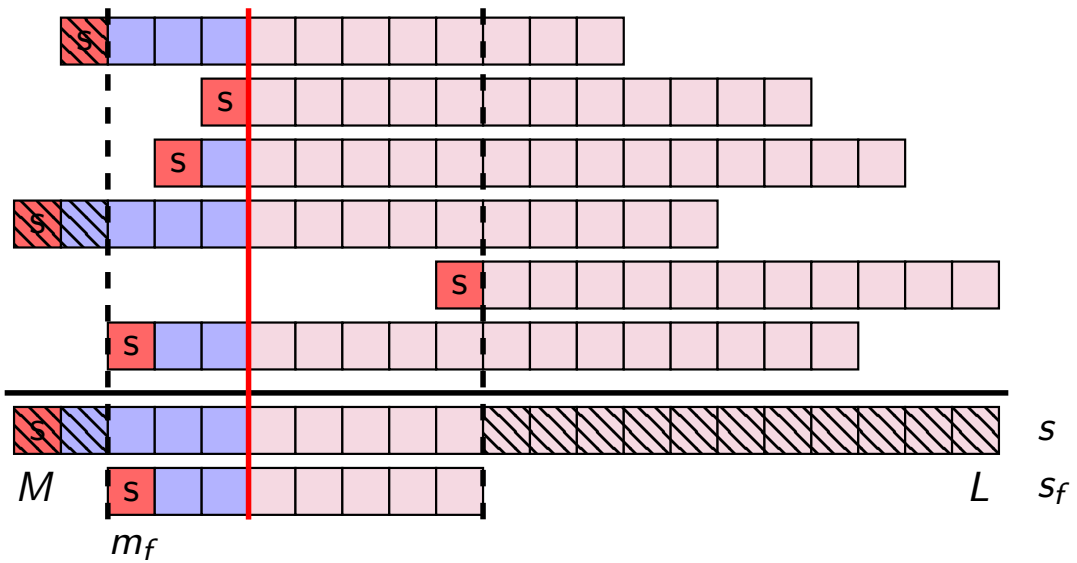
Example :

We want to compute $104 + 82 - 94$ with 8 bits :
$104 + 82 = -70$ overflow !
but $-70 - 94 = 92$ overflow !
This second overflow cancels the first one and we obtain the expected result.

Fixed-Point Jacskon's Rule

Let s be a sum of n fixed-point number p_is, in format (M, L). If s is known to have a final MSB equals to m_f with $m_f < M$, then:

$$s = \bigoplus_{1 \leq i \leq n}^{m_f+1} \left(\sum_{j=L}^{m_f} 2^j p_{i,j} \right)$$



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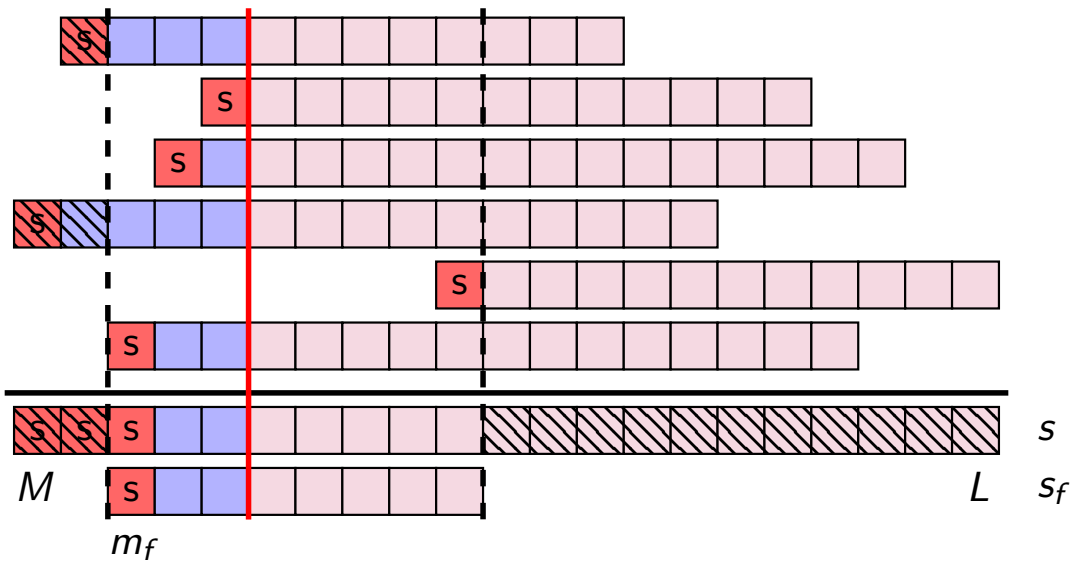
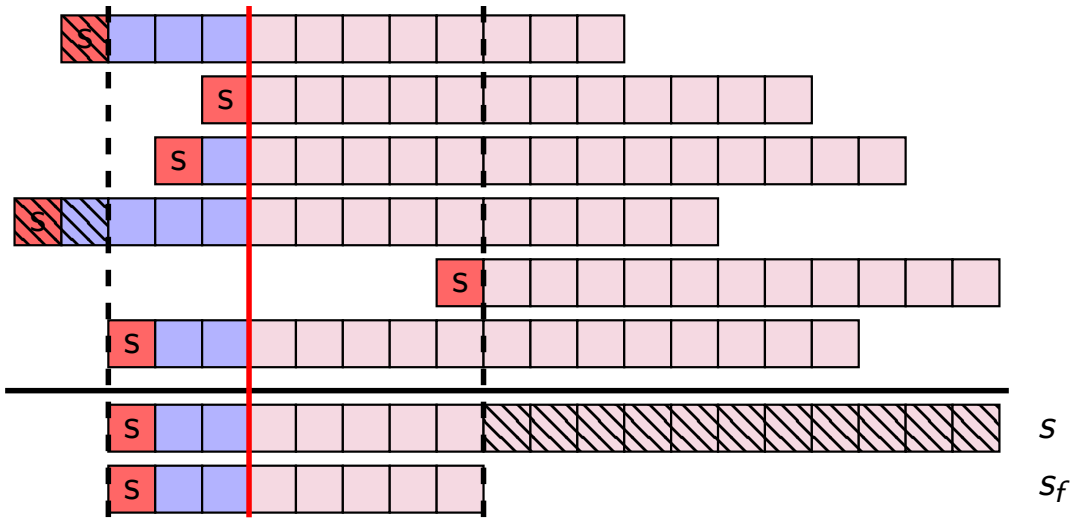


LSB Formatting main idea



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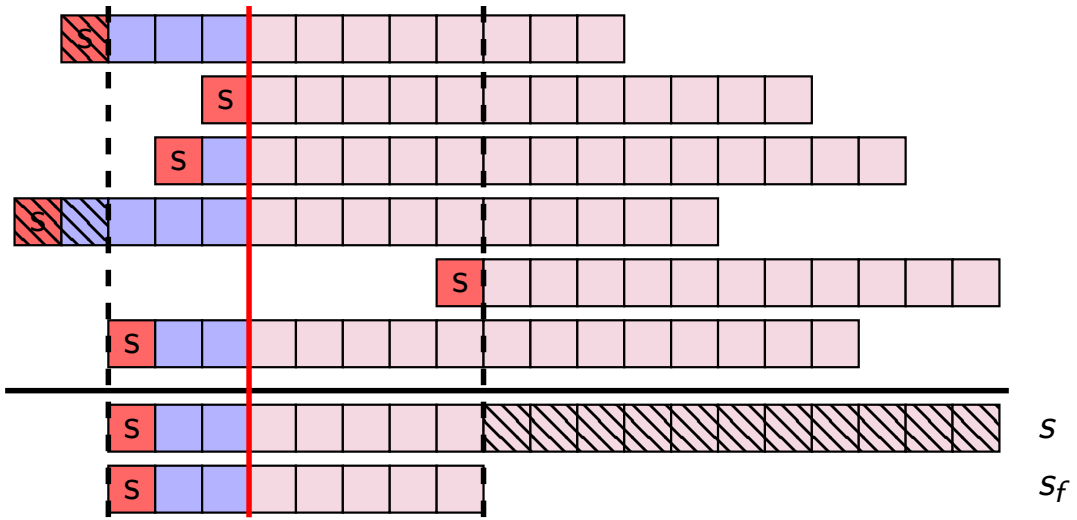

If we remove some bits, we will not compute s_f anymore but a faithful round-off of s_f can be acceptable.

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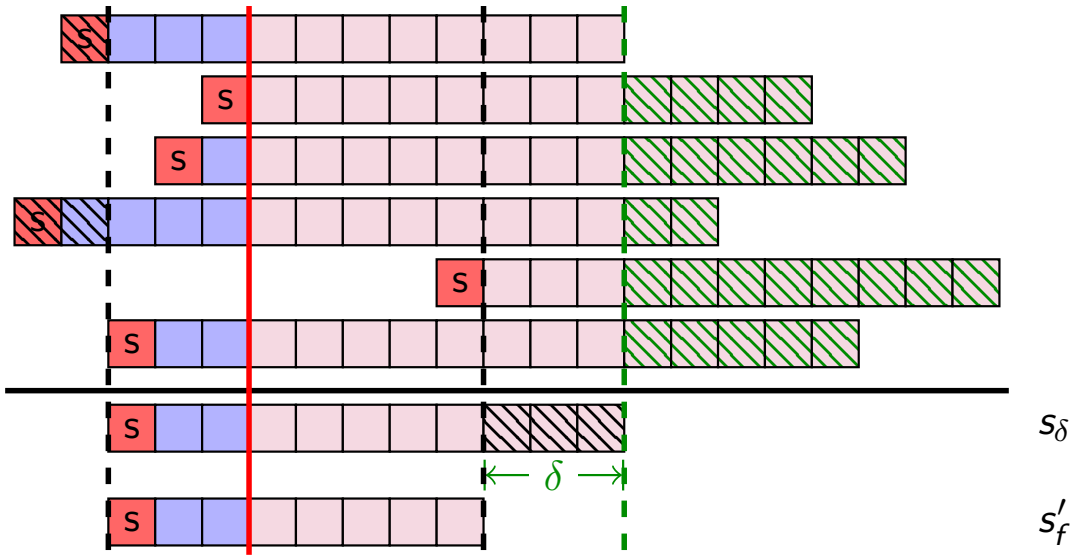

Can we determine a minimal δ such that s'_f is always a faithful round-off of s_f ?

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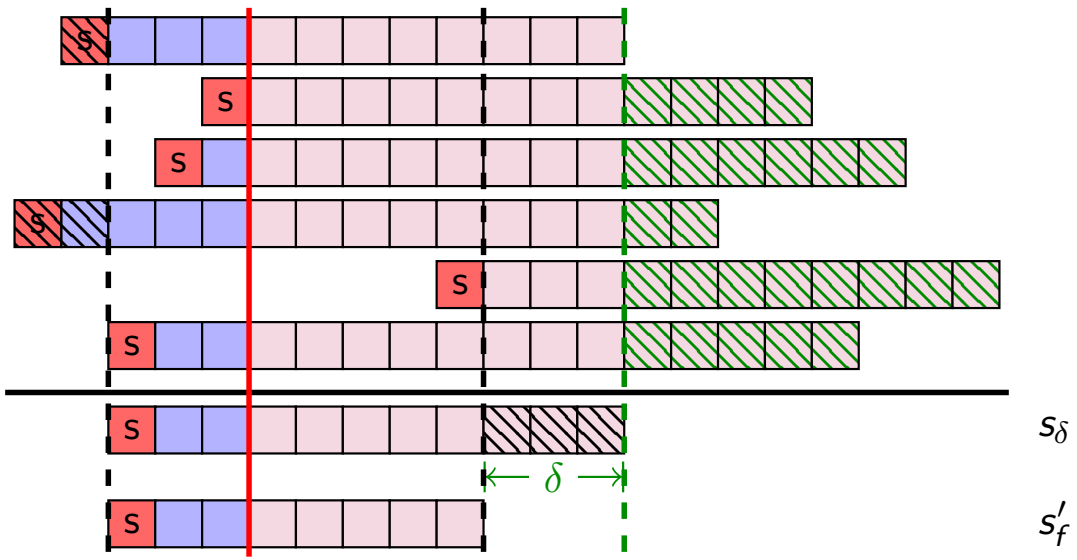
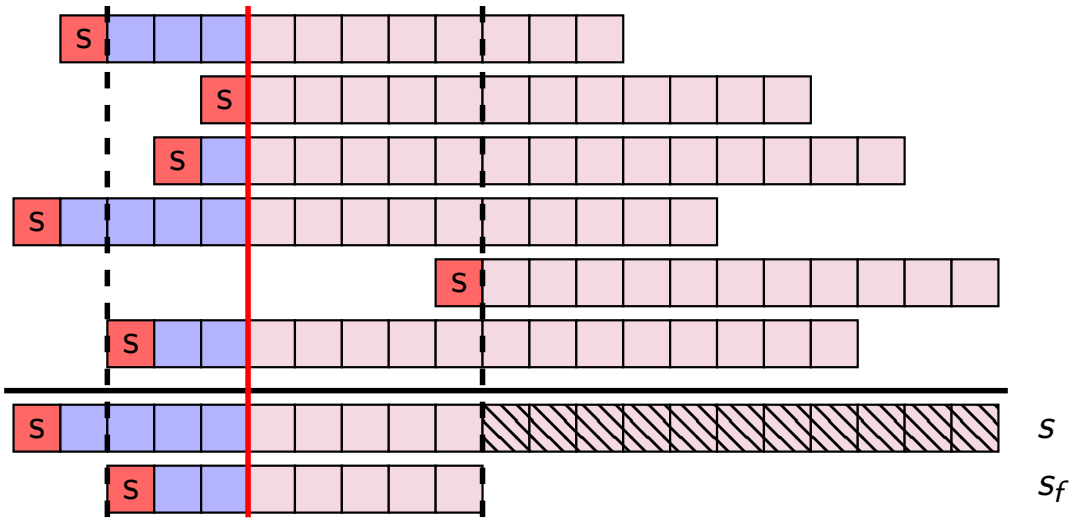


δ evaluation

For both rounding mode (round-to-nearest or truncation), the smallest integer δ that provides $s'_f = \star_{l_f}(s_f)$ is given by:

$$\delta = \lceil \log_2(n) \rceil$$

Formatting method



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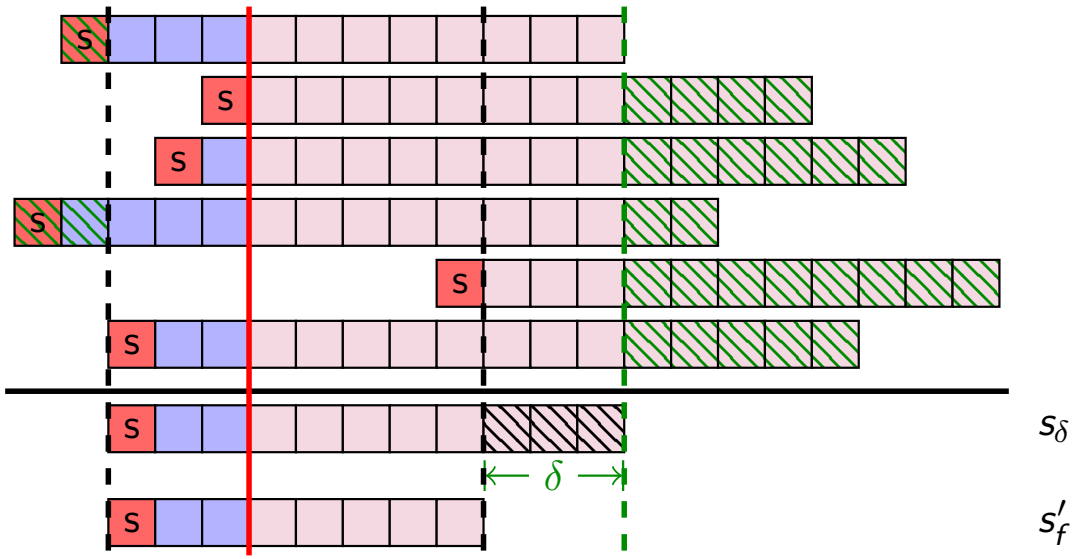

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2. we format all p_is into FPF $(m_f, l_f - \delta)$

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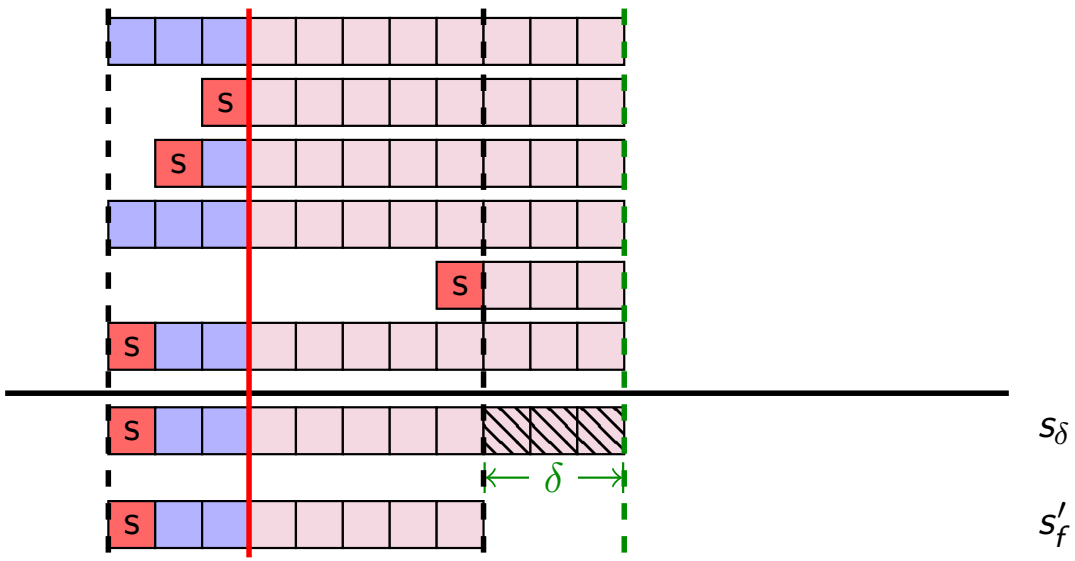

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3. we compute s_δ

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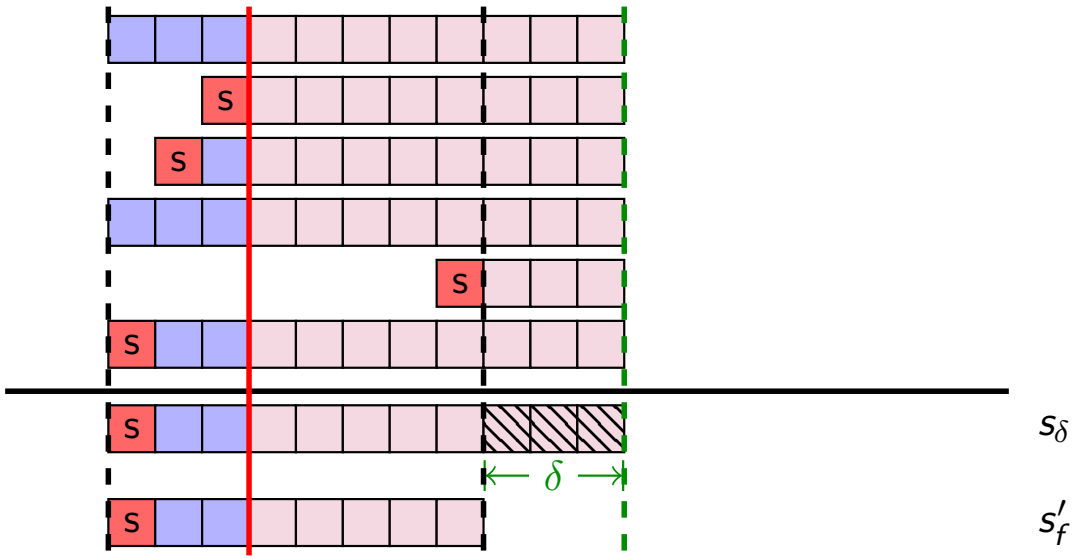

1. we compute δ
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4. we obtain s'_f from s_δ

Example

The following algorithm is the fixed-point algorithm of a $4^{th}-$order butterworth filter:

$$\begin{aligned} y(k) \;=\; & 0.0013279914856\ u(k) + 0.00531196594238\ u(k-1) \\ & +0.00796794891357\ u(k-2) + 0.00531196594238\ u(k-3) \\ & +0.0013279914856\ u(k-4) + 2.87109375\ y(k-1) \\ & -3.20825195312\ y(k-2) + 1.63458251953\ y(k-3) \\ & -0.318710327148\ y(k-4) \end{aligned}$$

Inputs datas :

- wordlength of constants, $u(k)$ and $y(k)$: 16 bits
- $u(k) \in [-13, 13]$

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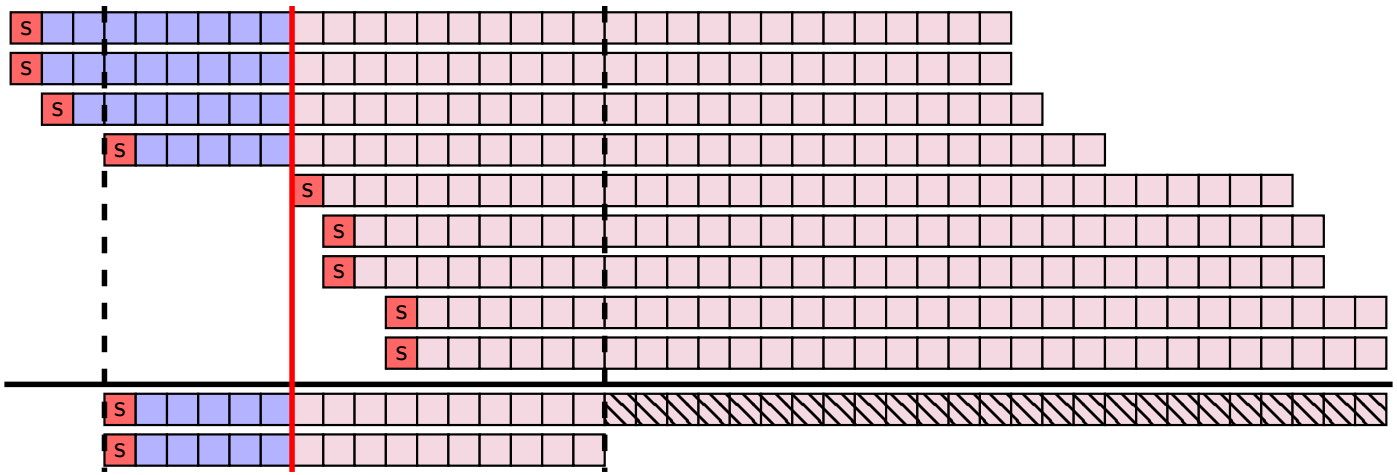

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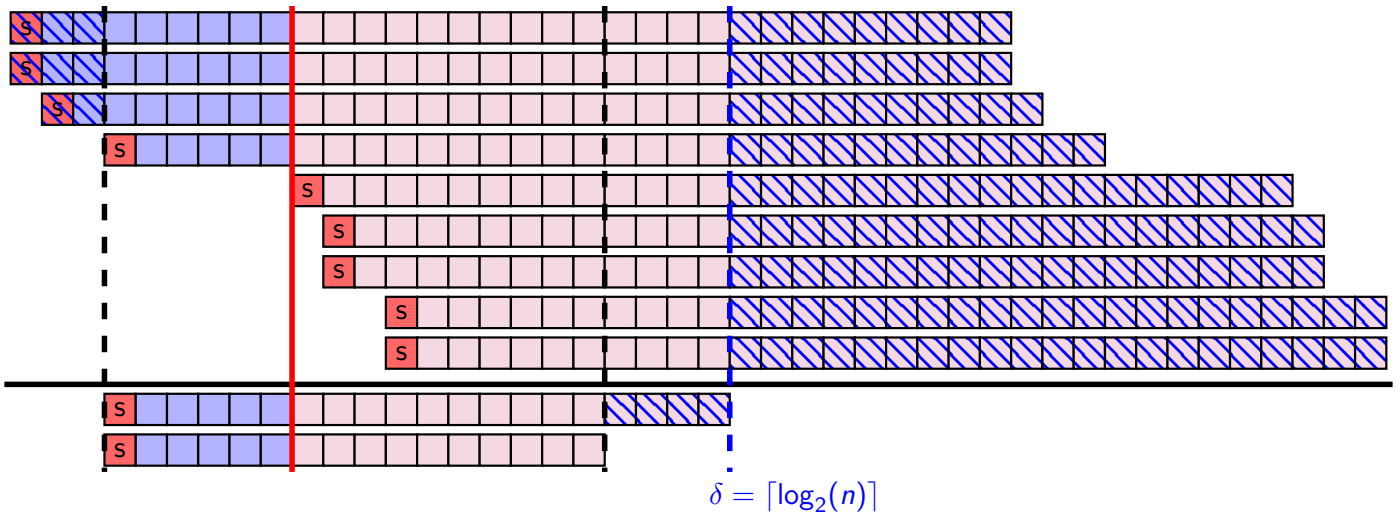

Conclusion

We try to answer the following question :

For a given sum-of-products, how to reduce the number of bits of the operands while controlling the error ?

For this, some works have been realized:

- FiPoGen : a tool generating fixed-point code for a given sum-of-products
- Bits formatting : a first step towards word-length optimization

THANK YOU

Any questions?