Reliable evaluation of the L_2-norm of a linear filter

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Context

Let H be a stable linear filter/controller with input/output relationship given by:

$$\left\{ \begin{array}{rcl} \boldsymbol{x}(k+1) & = & \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}\boldsymbol{u}(k) \\ \boldsymbol{y}(k) & = & \boldsymbol{C}\boldsymbol{x}(k) + \boldsymbol{D}\boldsymbol{u}(k) \end{array} \right.$$

where $\boldsymbol{u}(k)$, $\boldsymbol{x}(k)$ and $\boldsymbol{y}(k)$ are the input, state and output vectors of the filter at time k, and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ some matrices defining the filter.

The L_2-norm of this filter can be computed by

$$\|H\|_2 = \sqrt{tr\left(\boldsymbol{C}\boldsymbol{W}\boldsymbol{C}^\top + \boldsymbol{D}\boldsymbol{D}^\top\right)} \tag{1}$$

where $\boldsymbol{W}$ is the matrix solution of the discrete Lyapunov equation:

$$\boldsymbol{W} = \boldsymbol{A}\boldsymbol{W}\boldsymbol{A}^\top + \boldsymbol{B}\boldsymbol{B}^\top \tag{2}$$

or equivalently

$$\boldsymbol{W} = \sum_{k=0}^{\infty} \boldsymbol{A}^k \boldsymbol{B}\boldsymbol{B}^\top \boldsymbol{A}^{\top k}. \tag{3}$$

This measure is important in the context of the implementation of linear filters/controllers with finite word-length arithmetic, such as fixed-point or floating-point arithmetic. It is used to evaluate how finite precision modifies the filter.

Subject

The aim of this internship is to provide an efficient and reliable evaluation of this norm (*i.e.* provide a C or Python code evaluating this norm and the associated error analysis):

a) We will first use a naive method to solve the Lyapunov equation (2), solving a linear system with interval multi-precision arithmetic (using MPFI or Arb[1] library). Then the $\|H\|_2$ can be computed. The internal precision of the computation will be discussed (in order to obtain a reliable result up to a given ε).

b) Then, we will study the various efficient algorithms[2, 3] used to solve the Lyapunov equation (based on LU and Schur decomposition, etc.). One of them will be implemented with interval multi-precision arithmetic.

c) Finally, technics used in [4] can be used to compute the according to $\|H\|_2$ with equation (3) instead of (2). The two methods will be compared.

References

[1] F. Johansson. Arb: a C library for ball arithmetic. *ACM Communications in Computer Algebra*, 2013.

[2] V. Simoncini Computational Methods for Linear Matrix Equations SIAM Review, 2014.

[3] S. Hammarling. Numerical solution of the discrete-time, convergent, non-negative definite Lyapunov equation. In *Systems & Control Letters, 17*, 1991.

[4] Volkova, Anastasia and Hilaire, Thibault and Lauter, Christoph Q. Reliable evaluation of the Worst-Case Peak Gain matrix in multiple precision *22nd IEEE Symposium on Computer Arithmetic, Jun 2015, Lyon, France. 2015*